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EQUIVALENCE OF THREE TOPOLOGIES IN THE SPACES OF LAPLACE-STIELTJES INTEGRALS

For a non-negative nondecreasing unbounded continuous on the right function F and a real-valued function f on $(1, +\infty)$ the integral $I(\sigma) = \int_1^{\infty} f(x)e^{x\sigma}dF(x)$ is called the Laplace-Stieltjes integral. For some class of such integrals three various topologies are introduced and their equivalence is proven.

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Introduction

Let V be a class of a non-negative nondecreasing unbounded continuous on the right functions F on $(1, +\infty)$. We assume that a real-valued function f on $(1, +\infty)$ is such that the Lebesque-Stieltjes integral $\int_1^A |f(x)|e^{x\sigma}dF(x)$ exists for every $\sigma \in \mathbb{R}$ and $A \in (1, +\infty)$, and the integral

$$I(\sigma) = \int_1^{\infty} f(x)e^{x\sigma}dF(x), \quad \sigma \in \mathbb{R}, \tag{1}$$

is called of Laplace-Stieltjes [1]. We also remark that the Dirichlet series $I(\sigma) = \sum_{n=1}^{\infty} a_n e^{\lambda_n \sigma}$, $1 < \lambda_n \uparrow \infty$, can be rewritten in the form (1) with $f(x) = a_n$ for $x = \lambda_n$ and $f(x) = 0$ for $x \neq \lambda_n$ and $F(x) = n(x)$, where $n(x)$ is a counting function of (λ_n).

Let

$$M(\sigma) = M(\sigma, I) = \int_1^{\infty} |f(x)|e^{x\sigma}dF(x), \quad \sigma \in \mathbb{R}. \tag{2}$$

It is clear, that if $f(x) \geq 0$ for all $x \geq 0$ then $M(\sigma, I) = I(\sigma)$, and asymptotic properties of integrals of such kind are studied in a monograph [1]. As in [1, p.21] we say that a

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function $|f|$ has regular variation in regard to $F \in V$ if there exist $a \in [0, 1)$, $b \in [0, 1)$ and $h \in (0, +\infty) > 0$ such that for all $x > a$

$$\int\limits_{x-a}^{x+b} |f(t)| dF(t) \geq h|f(x)|. \tag{3}$$

By σ_M we denote an abscissa of the convergence of integral (2), i. e. integral (2) converges for $\sigma < \sigma_M$ and diverges for $\sigma > \sigma_M$. If integral (2) converges for all $\sigma \in \mathbb{R}$ then we put $\sigma_M = +\infty$. It is known [1, p. 21] that if $\ln F(x) = o(x)$ as $x \to +\infty$ and $|f|$ has regular variation in regard to $F \in V$ then $\sigma_M = +\infty$ if and only if

$$\frac{1}{x} \ln \frac{1}{|f(x)|} \to +\infty, \quad x \to +\infty, \tag{4}$$

i. e. $|f(x)| \leq e^{-Kx}$ for every $K > 0$ and all $x \geq x_0(K)$. Therefore, in [2] and [3] for a positive continuous on $[0, +\infty)$ function h increases to $+\infty$. By LS_h we denote a class of integrals I such that $|f(x)| \exp\{xh(x)\} \to 0$ as $x \to +\infty$ and define $||I||_h = \sup\{|f(x)| \exp\{xh(x)\} : x \geq 0\}$. For example, it is proven [3] that if $F \in V$ and $\ln F(x) = o(x)$ as $x \to +\infty$ then $(LS_h, || \cdot ||_h)$ is a non-uniformly convex Banach space.

In this article we will consider slightly different spaces of integrals (1).

1 Various topologies on $LS(U(F))$

For a fixed function $F \in V$ by $U(F)$ we denote a class of real function on $[1, +\infty)$ such that for every $f_1 \in U(F)$ and $f_2 \in U(F)$ the functions $|f_1|$, $|f_2|$ and $|f_1 - f_2|$ have the regular variation in regard to F, and by $LS(U(F))$ we denote a set of integrals (1) with $f \in U(F)$ and $\sigma_M = +\infty$.

At first we assume that (r_k) is a non-decreasing sequence of positive numbers, $r_k \to +\infty$ with k. If for each $I \in LS(U(F))$

$$||I||_{r_k} = \int\limits_{1}^{\infty} |f(x)| e^{xr_k} dF(x) \tag{5}$$

then $||I||_{r_k}$ exists for each k and it is easily seen that this is a norm on $LS(U(F))$. It is clear that $||I||_{r_k} \leq ||I||_{r_{k+1}}$ for all $k \geq 1$. With these countable norms $||I||_{r_k}$ $(k \geq 1)$ we define (see [4, p.37]) a metric topology on $I \in LS(U(F))$ with metric d:

$$d(I_1, I_2) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{||I_1 - I_2||_{r_k}}{1 + ||I_1 - I_2||_{r_k}}, \quad I_1, I_2 \in LS(F(U)). \tag{6}$$

Since $||I||_{r_k} \leq ||I||_{r_{k+1}}$ for all $k \geq 1$, it is clear that the metric topology defined by d is the sup topology which is locally convex (see [4, p. 33-37]).

We remark that there exist integrals (1) such that $M(\sigma, I) \equiv 0$ and $f(x) \not\equiv 0$. Indeed, these is for example if for all $n \in \mathbb{N}$

$$F(x) = \begin{cases} 0, \ 1 \leq x < 2, \\ n, \ 2n \leq x < 2(n+1) \end{cases}, \quad f(x) = \begin{cases} \alpha_n > 0, \ x = 2n - 1, \\ 0, \ x \neq 2n - 1 \end{cases}.$$

However the integral $M(\sigma, I) \in LS(U(F))$ is the additive zero of $LS(U(F))$ if and only if $|f(x)| = 0$ for each $x > 1$. Indeed, from the regular variation in regard to F we have for $\sigma = 0$ and each $x > 1$ (see (3))

$$0 = M(\sigma, I) = \int_1^\infty |f(x)| dF(x) \geq \int_{x-a}^{x+b} |f(t)| dF(t) \geq h|f(x)|,$$

i. e. $f(x) = 0$ for each $x > 1$.

Now for each $I \in LS(U(F))$ let

$$p(I) = \sup_{x>1} |f(x)|^{1/x} \tag{7}$$

and

$$||I||_{q_j} = \sup_{1 \leq x \leq q_j} |f(x)|^{1/x}, \tag{8}$$

where (q_j) is a non-decreasing sequence of positive numbers, $1 < q_j \to +\infty$ with j. As (7) is defined by the condition (4). Then the functions $p(f)$ and $||I||_{q_j}$ are paranorms on $LS(U(F))$ (see [5, p.85]).

We define metric topologies on $LS(U(F))$ with the metrics

$$p(I_1,\, I_2) = \sup_{x>1} |f_1(x) - f_2(x)|^{1/x}, \quad I_1,\, I_2 \in LS(U(F)),$$

and

$$s(I_1,\, I_2) = \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{||I_1 - I_2||_{q_j}}{1 + ||I_1 - I_2||_{q_j}}, \quad I_1,\, I_2 \in LS(U(F)).$$

Since $||I||_{q_j} \leq ||I||_{q_{j+1}}$, the topology s is the sup topology which is locally convex.

Theorem 1. *If*

$$\int_1^\infty e^{-tx} dF(x) \to 0, \quad t \to +\infty, \tag{9}$$

then the three topologies represented by d, p and s are equivalent.

Proof. First we show that the topologies given by d and p are equivalent. Let $I_m \in LS(U(F))$ and $I_m \to I \in LS(U(F))$ as $m \to \infty$ in the paranorm p.

Then if

$$I_m(\sigma) = \int_1^\infty f_m(x) e^{x\sigma} dF(x), \quad I(\sigma) = \int_1^\infty f(x) e^{x\sigma} dF(x),$$

we have $|f_m(x) - f(x)| \leq (1/c)^x$ for an arbitrarily large $c > 1$, all $m \geq m_0(c)$ and all $x > 1$. Therefore, due to (9) we have

$$||I_m - I||_{r_k} = \int_1^\infty |f_m(x) - f(x)| e^{xr_k} dF(x) \leq \int_1^\infty \exp\{-x(\ln\, c - r_k)\} dF(x) \to 0$$

as $c\to+\infty$, i. e. $I_m\to I$ as $m\to\infty$ under each norm $||I||_{r_k}$ and $I_m\to I$ as $m\to\infty$ in the metric d.

On the other hand, suppose that $I_m\to I$ as $m\to\infty$ in the metric d, that is under each norm $||I||_{r_k}$. Then for an arbitrarily large $c>1$, $m\ge m_0(c)$ and all $x>1$ we have in view of (3) with $|f_m(x)-f(x)|$ instead $f(x)$

$$\frac{1}{c}>\int\limits_1^{+\infty}|f_m(t)-f(t)|e^{tr_k}dF(t)\ge\int\limits_{x-a}^{x+b}|f_m(t)-f(t)|e^{tr_k}dF(t)\ge$$

$$\ge e^{(x-a)r_k}\int\limits_{x-a}^{x+b}|f_m(t)-f(t)|dF(t)\ge he^{(x-a)r_k}|f_m(x)-f(x)|, \tag{10}$$

i. e. $|f_m(x)-f(x)|<\dfrac{1}{hc}e^{-(x-a)r_k}\le\dfrac{1}{c^x}$, provided $r_k\ge\dfrac{1}{x-a}\ln\dfrac{c^{x-1}}{h}$. Thus, for all $m\ge m_0(c)$ all $x>1$ and all $k\ge k_0(c,x)$ we have $|f_m(x)-f(x)|^{1/x}<1/c$, i.e. $\sup\limits_{x\ge1}|f_m(x)-f(x)|^{1/x}\to0$ as $m\to\infty$ and $I_m\to I$ in the paranorm p. Hence it follows that the topologies given by d and p are equivalent.

To prove the other part of Theorem 1, let $I_m\to I$ as $m\to\infty$ in the paranorm p. Then $|f_m(x)-f(x)|\le(1/c)^x$ for an arbitrarily large c, all $m\ge m_0(c)$ and all $x>1$. Therefore,

$$||I_m-I||_{q_j}=\sup_{1\le x\le q_j}|f_m(x)-f(x)|^{1/x}\le1/c$$

and

$$\sum_{j=1}^{\infty}\frac{1}{2^j}\frac{||I_m-I||_{q_j}}{1+||I_m-I||_{q_j}}\le\sum_{j=1}^{\infty}\frac{1}{(c+1)2^j}\to0,\quad c\to\infty,$$

i. e. $I_m\to I$ in the paranorm s.

On the other hand, if $I_m\to I$ in the paranorm s then $||I_m-I||_{q_j}\to0$ as $m\to\infty$ for each q_j and, thus, $|f_m(x)-f(x)|^{1/x}\le1/c$ for an arbitrarily large c, all $m\ge m_0(c)$, all $x\in(1,\,q_j]$ and all q_j, that is for all $x>1$. Hence $|f_m(x)-f(x)|^{1/x}\to0$ as $m\to\infty$ for all $x>1$ and, therefore, $I_m\to I$ in the paranorm p. Thus, the topologies given by p and s are equivalent. Theorem 1 is proved. □

The following theorem establishes a connection between the convergence under $p(I)$ and the convergence on every finite interval.

Theorem 2. *If F satisfies* (9) *and a sequence $(I_m)\subset LS(U(F))$ converges to $I\in LS(U(F))$ under $p(I)$ then $I_m\to I$ uniformly converges on every finite interval.*

Proof. Let $p(I_m-I)\to0$ as $m\to\infty$. Then for a given $\varepsilon>0$, there exists an $m_0=m_0(\varepsilon)$ such that $|f_m(x)-f(x)|\le\varepsilon^x$ for $m\ge m_0$, or $|f_m(x)-f(x)|\le\varepsilon^{-nx}$, where $n\in\mathbb{N}$ is arbitrarily large. Therefore, for $m\ge m_0$ and $\sigma\in[\sigma_1,\,\sigma_2]$ due to (9) we have

$$|I_m(\sigma)-I(\sigma)|<\int\limits_1^{+\infty}e^{x(\sigma_2-n)}dF(x)\to0,\quad n\to\infty.$$

Theorem 2 is proved. □

Remark 1. An opposite statement to Theorem 2 does not hold. Indeed, let for every $m \in \mathbb{Z}_+$ and $n \in \mathbb{N}$

$$F(x) = \begin{cases} 0, \ 1 \le x < 2, \\ n, \ 2n \le x < 2(n+1) \end{cases}, \quad f_m(x) = \begin{cases} \alpha_{m,n} > 0, \ x = 2n-1, \\ 0, \ x \ne 2n-1 \end{cases}.$$

Then for all $m \in \mathbb{Z}_+$

$$I_m(\sigma) = \int\limits_0^\infty f_m(x) e^{x\sigma} dF(x) = \sum_n f_m(2n) e^{2n\sigma} = 0,$$

i. e. $I_m(\sigma) \to I_0(\sigma)$ for all $\sigma \in [\sigma_1, \sigma_2]$. On the other hand,

$$p(I_m, I_0) = \sup\left\{|f_m(x) - f_0(x)|^{1/x} : x > 1\right\} \ge$$
$$\ge |f_m(3) - f_0(3)|^{1/3} = |\alpha_{m,3} - \alpha_{0,3}|^{1/3} \ge h_1 > 0$$

provided $\alpha_{m,3} - \alpha_{0,3} \ge \eta_1 > 0$ for all $m \in \mathbb{N}$.

2 Completeness.

Now we will show the completeness of the space $LS(U(F))$ under various topologies established above. Note that it is sufficient to establish the completeness under one of them, due to Theorem 1.

Theorem 3. *If F satiesfies (10) then the space $(LS(U(F)), s)$ is complete.*

Proof. Let (I_ν) be a s-Cauchy sequence on $LS(U(F))$. Then for a given $\varepsilon \in (0, 1)$ there exists a $Q = Q(\varepsilon)$ such that for all $\nu \ge Q$ and $n \ge Q$

$$\sum_{j=1}^\infty \frac{1}{2^j} \frac{||I_\nu - I_n||_{q_j}}{1 + ||I_\nu - I_n||_{q_j}} < \varepsilon,$$

whence

$$|f_\nu(x) - f_n(x)|^{1/x} < \varepsilon_1, \quad \nu \ge Q, n \ge Q, 1 \le x \le q_j,$$

where $\varepsilon_1 = \varepsilon_1(\varepsilon) \to 0$ as $\varepsilon \to 0$. Hence the sequence (f_ν) in p tends to $f(x)$ for each $x > 1$. Since $\dfrac{1}{x} \ln \dfrac{1}{|f_\nu(x)|} \to +\infty$ as $x \to +\infty$, we have $f_\nu(x) \le \varepsilon_1^x$ for $x \ge x_0(\varepsilon_1)$ and, therefore,

$$|f(x)| \le |f_\nu(x) - f(x)| + f|_\nu(x)| \le 2\varepsilon_1^x, \quad x \ge x_0(\varepsilon_1).$$

Thus, (4) holds for f and $I^*(\sigma) = \int\limits_1^\infty |f(x)| e^{x\sigma} dF(x)$ has the abscissa of the convergence $\sigma_M = +\infty$.

Since all $f_\nu \in U$, the function $f \in U$. Indeed, if $|f|$ does not have a regular variation in regard to $F \in V$ then for all $a \ge 0$, all $b \ge 0$ and all $h > 0$ there exists $x* > 1 + a$ such that

$$\int\limits_{x^*-a}^{x^*+b} |f(t)| dF(t) < h|f(x^*)|. \tag{11}$$

Clearly, $|f(x^*)| > 0$ and, therefore, $f_\nu(x^*) > 0$ for $\nu \geq \nu_0$. Since f_ν has regular variation in regard to $F \in V$, there exist $a \in [0,1)$, $b \in [0,1)$ and $h_1 > 0$ such that

$$\int\limits_{x^*-a}^{x^*+b} |f_\nu(t)| dF(t) \geq h_1 |f_\nu(x^*)|. \tag{12}$$

But $f_\nu(x^*) \to f(x^*)$ as $\nu \to \infty$. Therefore, from (11) and (12) we obtain for $\nu \to \infty$

$$h_1|f(x^*)| + o(1) = h_1|f_\nu(x^*)| \leq \int\limits_{x^*-a}^{x^*+b} |f_\nu(t)| dF(t)$$
$$= \int\limits_{x^*-a}^{x^*+b} |f(t)| dF(t) + o(1) < h|f(x^*)| + o(1).$$

This is impossible of the arbitrariness of h. Thus, $I \in LS(U(F))$.

Finally,

$$||I_\nu - I||_{q_j} = \sup_{1 < x \leq q_j} |f_\nu(x) - f(x)|^{1/x} < \varepsilon$$

for all $\nu \geq Q$. Hence

$$s(I_\nu,\, I) = \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{||I_\nu - I||_{q_j}}{1 + ||I_\nu - I||_{q_j}} < \frac{\varepsilon}{1+\varepsilon}, \quad \nu \geq Q,$$

i. e. $(LS(U(F)), s)$ is complete. Theorem 3 is proved. □

Corollary 1. *If F satisfies (9) then the spaces $(LS(U(F)), s)$, $(LS(U(F)), p)$, $(LS(U(F)), d)$ and the space $LS(U(F))$ endowed with the compact open topology (as in Theorem 2) are Frechet spaces and, thus, are barrelled spaces.*

Theorem 4. *If F satiesfies (9) then the space $(LS(U(F)), p)$ is a Montel space (see [6, p.32]).*

Proof. Let X be an arbitrary uniformly bounded subset of $(LS(U(F)), p)$, i. e. there exists $C \in (1, +\infty)$ such that $p(I) \leq C$ for all $I \in X$. For $I \in X$ as from (7) we have $|f(x)| \leq C^x$ for all $x > 1$ and, as above, $|f(x)| \leq e^{-nx}$ for each $n > 1$ and all $x \geq x_0 = x_0(n)$. Therefore, for $\sigma \in D := [\sigma_1,\, \sigma_2]$ we have

$$|I'(\sigma)| \leq \int\limits_1^{+\infty} x|f(x)| e^{x\sigma_2} dF(x) \leq \left(\int\limits_1^{x_0} + \int\limits_{x_0}^{\infty} \right) |f(x)| e^{x(\sigma_2+1)} dF(x) \leq$$

$$\leq \max\{C^x e^{x(\sigma_2+1)} :\, 1 \leq x \leq x_0\} \int\limits_1^{x_0} dF(x) + \int\limits_{x_0}^{\infty} e^{-(n-\sigma_2-1)x} dF(x).$$

Hence from the arbitrariness of n and (9) we have $|I'(\sigma)| \le C_D$ for all $I \in X$ and $\sigma \in D$, where C_D is a constant depending on D, and for $\sigma', \sigma'' \in D$, $\sigma' < \sigma''$, we obtain for some $\xi \in [\sigma', \sigma'']$

$$|I(\sigma'') - I(\sigma')| = \int\limits_1^{+\infty} x|f(x)|e^{x\xi} dF(x) \le C_D^*(\sigma'' - \sigma')$$

for all $I \in X$, i. e. X is equi-continuos. Now by a well-known argument we can select a subsequence of X which converges uniformly on D to a function I.From the arbitrariness of D and the completeness of $(LS(U(F)), p)$ Theorem 4 is proved. □

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Для невід'ємної, неспадної, необмеженої справа, неперервної функції F та дійснозначної функції f, заданої на $(1, +\infty)$, інтеграл $I(\sigma) = \int_1^{\infty} f(x)e^{x\sigma} dF(x)$ називається інтегралом Лапласа-Стілтьєса. Для певного класу таких інтегралів введено три топології та доведено їхню еквівалентність.