

SHEREMETA M.M.

HADAMARD COMPOSITION OF SERIES IN SYSTEMS OF FUNCTIONS

For regularly converging in $\mathbb{C}$ series $A_j(z) = \sum_{n=1}^{\infty} a_{n,j} f(\lambda_n z)$, $1 \leq j \leq p$, where f is an entire transcendental function, the asymptotic behavior of a Hadamard composition $A(z) = (A_1 * \dots * A_p)_m(z) = \sum_{n=1}^{\infty} \left(\sum_{k_1 + \dots + k_p = m} c_{k_1 \dots k_p} a_{n,1}^{k_1} \cdot \dots \cdot a_{n,p}^{k_p} \right) f(\lambda_n z)$ of genus m is investigated. The function A_1 is called dominant, if $|c_{m0\dots 0}| |a_{n,1}|^m \neq 0$ and $|a_{n,j}| = o(|a_{n,1}|)$ as $n \rightarrow \infty$ for $2 \leq j \leq p$. The generalized order of a function A_j is called the quantity $\varrho_{\alpha,\beta}[A_j] = \overline{\lim}_{r \rightarrow +\infty} \frac{\alpha(\ln \mathfrak{M}(r, A_j))}{\beta(\ln r)}$, where $\mathfrak{M}(r, A_j) = \sum_{n=1}^{\infty} |a_{n,j}| M_f(r \lambda_n)$, $M_f(r) = \max\{|f(z)| : |z| = r\}$ and the functions α and β are positive, continuous and increasing to $+\infty$.

Under certain conditions on α , β , $M_f(r)$ and (λ_n) , it is proved that if among the functions A_j there exists a dominant one, then $\varrho_{\alpha,\beta}[A] = \max\{\varrho_{\alpha,\beta}[A_j] : 1 \leq j \leq p\}$. In terms of generalized orders, a connection is established between the growth of the maximal terms of power expansions of the functions $(A_1^{(k)} * \dots * A_p^{(k)})_m$ and $((A_1 * \dots * A_p)_m)^{(k)}$. Unresolved problems are formulated.

Key words and phrases: entire function, regularly converging series, Hadamard composition, generalized order.

Ivan Franko National University of Lviv, str. Universytetska, 1, Lviv, 79000, Ukraine
e-mail: *m.m.sheremeta@gmail.com*

INTRODUCTION

Let

$$f(z) = \sum_{k=0}^{\infty} f_k z^k \quad (1)$$

be an entire transcendental function, $M_f(r) = \max\{|f(z)| : |z| = r\}$ and (λ_n) be a sequence of positive numbers increasing to $+\infty$. Suppose that the series

$$A(z) = \sum_{n=1}^{\infty} a_n f(\lambda_n z) \quad (2)$$

УДК 517.547.2

2010 *Mathematics Subject Classification:* 30B50, 30D10, 30D20.

Information on some grant ...

in the system $f(\lambda_n z)$ regularly converges in $\mathbb{C}$, i. e. for all $r \in [0, +\infty)$

$$\mathfrak{M}(r, A) := \sum_{n=1}^{\infty} |a_n| M_f(r \lambda_n) < +\infty. \quad (3)$$

Many authors have studied the representation of analytic functions by series in the system $f(\lambda_n z)$. We will specify here only on the monographs of A.F. Leont'ev [5] and B.V. Vynnytskyi [12], where references to other works can be find.

Since series (2) regularly converges in $\mathbb{C}$, the function A is entire. Generalized orders are used to study its growth. For this purpose, as in [7] by L we denote a class of continuous non-negative on $(-\infty, +\infty)$ functions α such that $\alpha(x) = \alpha(x_0) \geq 0$ for $x \leq x_0$ and $\alpha(x) \uparrow +\infty$ as $x_0 \leq x \rightarrow +\infty$. We say that $\alpha \in L^0$, if $\alpha \in L$ and $\alpha((1+o(1))x) = (1+o(1))\alpha(x)$ as $x \rightarrow +\infty$. Finally, $\alpha \in L_{si}$, if $\alpha \in L$ and $\alpha(cx) = (1+o(1))\alpha(x)$ as $x \rightarrow +\infty$ for each $c \in (0, +\infty)$, i. e. α is a slowly increasing function. Clearly, $L_{si} \subset L^0$. For $\alpha \in L$ and $\beta \in L$ quantity $\varrho_{\alpha, \beta}[f] = \overline{\lim}_{r \rightarrow +\infty} \frac{\alpha(\ln M_f(r))}{\beta(\ln r)}$ is called [7] generalized (α, β) -order of the entire function f .

In the papers [11, 10] the relationship between the growth of functions $M_f(r)$, $\mathfrak{M}(r, A)$ and $M_f^{-1}(\mathfrak{M}(r, A))$ was studied. When studying the logarithmic convexity of the function $\ln M_f(r)$ was used, from which it follows that

$$\Gamma_f(r) := \frac{d \ln M_f(r)}{d \ln r} \nearrow +\infty, \quad r \rightarrow +\infty,$$

(in points where the derivative does not exist, under $\frac{d \ln M_f(r)}{d \ln r}$ we mean right-hand derivative). For example, in [10] the following theorem was proved.

Theorem. Let $\alpha \in L_{si}$, $\beta \in L^0$, $\frac{d\beta^{-1}(c\alpha(x))}{d \ln x} = O(1)$ as $x \rightarrow +\infty$ for each $c \in (0, +\infty)$ and

$$\varrho_{\alpha, \beta}[A] := \varrho_{\alpha, \beta}[\mathfrak{M}] = \overline{\lim}_{r \rightarrow +\infty} \frac{\alpha(\ln \mathfrak{M}(r, A))}{\beta(\ln r)}.$$

If $a_n \geq 0$ for all $n \geq 1$, series (2) regularly converges in $\mathbb{C}$, $\ln n = O(\Gamma_f(\lambda_n))$ and $\ln \lambda_n = o\left(\beta^{-1}(c\alpha\left(\frac{1}{\ln \lambda_n} \ln \frac{1}{a_n}\right))\right)$ as $n \rightarrow \infty$ each $c \in (0, +\infty)$ then $\varrho_{\alpha, \beta}[A] = \varrho_{\alpha, \beta}[f]$.

Let $G_j(z) = \sum_{n=0}^{\infty} g_{n,j} z^n$, $1 \leq j \leq p$, be an entire transcendental functions. As in [6], we say that the function $G(z) = \sum_{n=0}^{\infty} g_n z^n$ is similar to the Hadamard composition of the functions g_j if $g_n = w(g_{n,1}, \dots, g_{n,p})$ for all n , where $w : \mathbb{C}^p \rightarrow \mathbb{C}$ is a continuous function. Clearly, if $p = 2$ and $w(g_{n,1}, g_{n,2}) = g_{n,1} g_{n,2}$ then $g = (g_1 * g_2)$ is [4] the Hadamard composition (product) of the functions g_1 and g_2 . Properties of this composition obtained by J.Hadamard find the applications [3, 1] in the theory of the analytic continuation of the functions represented by power series.

The article [9] considers the case when w is a homogeneous polynomial. Recall that a polynomial is named homogeneous if all monomials with nonzero coefficients have the

identical degree. A polynomial $P(x_1, \dots, x_p)$ is homogeneous of the degree m if and only if $P(tx_1, \dots, tx_p) = t^m P(x_1, \dots, x_p)$ for all t from the field above that a polynomial is defined. Function (1) is called a Hadamard composition of genus $m \geq 1$ of functions (2) if $a_n = P(a_{n,1}, \dots, a_{n,p})$, where

$$P(x_1, \dots, x_p) = \sum_{k_1 + \dots + k_p = m} c_{k_1 \dots k_p} x_1^{k_1} \cdot \dots \cdot x_p^{k_p}, \quad k_j \in \mathbb{Z}_+, \quad (4)$$

is a homogeneous polynomial of degree $m \geq 1$ with constant coefficients $c_{k_1 \dots k_p}$. We remark that the usual Hadamard composition is a special case of the Hadamard composition of the genus $m = 2$. Hadamard composition of genus $m \geq 1$ of functions f_j is denoted by

$$(G_1 * \dots * G_p)_m, \text{ i. e. } (G_1 * \dots * G_p)_m(z) = \sum_{n=0}^{\infty} g_n z^n = \sum_{n=0}^{\infty} \left(\sum_{k_1 + \dots + k_p = m} c_{k_1 \dots k_p} g_{n,1}^{k_1} \cdot \dots \cdot g_{n,p}^{k_p} \right) z^n.$$

Here we study the properties of Hadamard compositions of genus $m \geq 1$ for entire functions represented by series in a system of functions.

1 DEFINITION AND CONVERGENCE

Function (2) is called a Hadamard composition of genus m of the functions

$$A_j(z) = \sum_{n=1}^{\infty} a_{n,j} f(\lambda_n z), \quad 1 \leq j \leq p, \quad (5)$$

if $a_n = P(a_{n,1}, \dots, a_{n,p})$, where P is defined by (4). Then as above

$$A(z) = (A_1 * \dots * A_p)_m(z) = \sum_{n=1}^{\infty} a_n f(\lambda_n z) = \sum_{n=1}^{\infty} \left(\sum_{k_1 + \dots + k_p = m} c_{k_1 \dots k_p} a_{n,1}^{k_1} \cdot \dots \cdot a_{n,p}^{k_p} \right) f(\lambda_n z). \quad (6)$$

At first we prove the following lemma.

Lemma 1. *Let $\ln n = o(\Gamma_f(\lambda_n))$ as $n \rightarrow \infty$. Then series (2) regularly converges in $\mathbb{C}$ if and only if*

$$\lim_{n \rightarrow \infty} \frac{1}{\lambda_n} M_f^{-1} \left(\frac{1}{|a_n|} \right) = +\infty. \quad (7)$$

Proof. If series (2) regularly converges in $\mathbb{C}$ then $|a_n| M_f(r \lambda_n) \rightarrow 0$ as $n \rightarrow \infty$, i. e. $|a_n| M_f(r \lambda_n) \leq 1$ for all $r \in [0, +\infty)$ and $n \geq n_0(r)$, whence $\frac{1}{\lambda_n} M_f^{-1} \left(\frac{1}{|a_n|} \right) \geq r$ for all $n \geq n_0(r)$. In view of the arbitrariness of r we get (7).

On the other hand, if $r \in [1, +\infty)$ is an arbitrary number and (7) holds then for every $K > r$ and all $n \geq n_0 = n_0(K)$ we have $\frac{1}{\lambda_n} M_f^{-1} \left(\frac{1}{|a_n|} \right) \geq K$, i. e. $|a_n| M_f(K \lambda_n) \leq 1$.

Therefore,

$$\begin{aligned}
\sum_{n=n_0}^{\infty} |a_n| M_f(r\lambda_n) &= \sum_{n=n_0}^{\infty} |a_n| M_f(K\lambda_n) \frac{M_f(r\lambda_n)}{M_f(K\lambda_n)} \leq \sum_{n=n_0}^{\infty} \frac{M_f(r\lambda_n)}{M_f(K\lambda_n)} \\
&= \sum_{n=n_0}^{\infty} \exp \left\{ - \int_{r\lambda_n}^{K\lambda_n} \frac{d \ln M_f(t)}{d \ln t} d \ln t \right\} = \sum_{n=n_0}^{\infty} \exp \left\{ - \int_{r\lambda_n}^{K\lambda_n} \Gamma_f(t) d \ln t \right\} \\
&\leq \sum_{n=n_0}^{\infty} \exp \{ -\Gamma_f(r\lambda_n) \ln(K/r) \} \leq \sum_{n=n_0}^{\infty} \exp \{ -\Gamma_f(\lambda_n) \ln(K/r) \} < +\infty,
\end{aligned}$$

because $\ln n = o(\Gamma_f(\lambda_n))$ as $n \rightarrow \infty$. Lemma 1 is proved. $\square$

We say that the function A_1 is dominant, if $|c_{m0\dots 0}| |a_{n,1}|^m \neq 0$ and $|a_{n,j}| = o(|a_{n,1}|)$ as $n \rightarrow \infty$ for $2 \leq j \leq p$. Using Lemma 1 we prove the following theorem.

Theorem 1. *Let $\ln n = o(\Gamma_f(\lambda_n))$ as $n \rightarrow \infty$. If all series (5) regularly converge in $\mathbb{C}$ then their Hadamard composition of genus $m \geq 1$ regularly converges in $\mathbb{C}$.*

If the function A_1 is dominant and Hadamard composition of genus $m = 1$ of the functions A_j regularly converges in $\mathbb{C}$ then each series (5) regularly converges in $\mathbb{C}$.

Proof. At first we remark that the condition $\frac{d \ln M_f(r)}{d \ln r} \nearrow +\infty$ as $r \rightarrow +\infty$ implies the condition $\frac{d \ln M_f^{-1}(x)}{d \ln x} \searrow +\infty$ as $x \rightarrow +\infty$, i. e. the function M_f^{-1} is slowly increasing.

Since series (2) is Hadamard composition of genus $m \geq 1$ of regularly convergent in $\mathbb{C}$ series (5) then

$$|a_n| \leq \sum_{k_1+\dots+k_p=m} |c_{k_1\dots k_p}| |a_{n,1}|^{k_1} \cdot \dots \cdot |a_{n,p}|^{k_p} \quad (8)$$

and by Lemma 1 $|a_{n,j}| \leq 1/M_f(K\lambda_n)$ for every $K > 0$, all $1 \leq j \leq p$ and all $n \geq n_0 = n_0(K)$. Therefore,

$$|a_n| \leq \sum_{k_1+\dots+k_p=m} |c_{k_1\dots k_p}| \left(\frac{1}{M_f(K\lambda_n)} \right)^{k_1} \cdot \dots \cdot \left(\frac{1}{M_f(K\lambda_n)} \right)^{k_p} = C \left(\frac{1}{M_f(K\lambda_n)} \right)^m,$$

where $C = \sum_{k_1+\dots+k_p=m} |c_{k_1\dots k_p}|$, and thus,

$$\begin{aligned}
\frac{1}{\lambda_n} M_f^{-1} \left(\frac{1}{|a_n|} \right) &\geq \frac{1}{\lambda_n} M_f^{-1} \left(\frac{M_f^m(K\lambda_n)}{C} \right) = \frac{1+o(1)}{\lambda_n} M_f^{-1}(M_f^m(K\lambda_n)) \\
&\geq \frac{1+o(1)}{\lambda_n} M_f^{-1}(M_f(K\lambda_n)) = (1+o(1))K, \quad n \rightarrow \infty,
\end{aligned}$$

whence in view of the arbitrariness of K (7) follows, i. e. series (2) regularly converges in $\mathbb{C}$

Now we suppose the function A_1 is dominant and $m \geq 1$. We put

$$\begin{aligned}
\Sigma'_n &= \sum_{k_1+\dots+k_p=m} c_{k_1\dots k_p} (a_{n,1})^{k_1} \cdot \dots \cdot (a_{n,p})^{k_p} \\
&= \sum_{k_1+\dots+k_p=m} c_{k_1\dots k_p} (a_{n,1})^{k_1} \cdot \dots \cdot (a_{n,p})^{k_p} - c_{m0\dots 0} (a_{n,1})^m.
\end{aligned}$$

Since for each monomial of the polynomial Σ'_n the sum of the exponents is equal to m , we have

$$\frac{|a_{n,1}|^{k_1} \cdot \dots \cdot |a_{n,p}|^{k_p}}{|a_{n,1}|^m} = \frac{|a_{n,2}|^{k_2} \cdot \dots \cdot |a_{n,p}|^{k_p}}{|a_{n,1}|^{m-k_1}} \rightarrow 0, \quad n \rightarrow \infty$$

and, thus $\Sigma'_n = o(|a_{n,1}|^m)$ as $n \rightarrow \infty$. Therefore,

$$|a_n| \geq |c_{m0\dots 0}| |a_{n,1}|^m - |\Sigma'_n| = |c_{m0\dots 0}| |a_{n,1}|^m - o(|a_{n,1}|^m) \geq \frac{|c_{m0\dots 0}|}{2} |a_{n,1}|^m, \quad n \geq n_0^*$$

i. e. $|a_{n,1}| \leq c|a_n|$ for $n \geq n_0^*$ provided $m = 1$, where $c = \text{const} > 0$. Thus, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{\lambda_n} M_f^{-1} \left(\frac{1}{|a_{n,j}|} \right) &\geq \lim_{n \rightarrow \infty} \frac{1}{\lambda_n} M_f^{-1} \left(\frac{1}{|a_{n,1}|} \right) \geq \lim_{n \rightarrow \infty} \frac{1}{\lambda_n} M_f^{-1} \left(\frac{1}{c|a_n|} \right) \\ &= \lim_{n \rightarrow \infty} \frac{1}{\lambda_n} M_f^{-1} \left(\frac{1}{|a_n|} \right) = +\infty, \end{aligned}$$

i. e. all series (6) regularly converge in $\mathbb{C}$. Theorem 1 is proved. $\square$

2 MAXIMAL TERM AND CENTRAL INDEX

Let series (2) regularly converges in $\mathbb{C}$, $\mu(r, A) = \max\{|a_n| M_f(r\lambda_n) : n \geq 1\}$ be the maximal term and $\nu(r, A) = \max\{n \geq 1 : |a_n| M_f(r\lambda_n) = \mu(r, A)\}$ be the central index of series (3). The following lemma is true.

Lemma 2. *The functions $\ln \mu(r, A)$, $\lambda_{\nu(r, A)}$ and $\nu(r, A)$ are non-decreasing and*

$$\ln \mu(r, A) - \ln \mu(r_0, A) = \int_{r_0}^r \frac{\Gamma_f(t\lambda_{\nu(t, A)})}{t} dt, \quad 0 \leq r_0 \leq r < +\infty. \quad (9)$$

Proof. For $h > 0$ we have

$$\begin{aligned} \mu(r+h, A) &= |a_{\nu(r+h, A)}| M_f((r+h)\lambda_{\nu(r+h, A)}) \\ &= |a_{\nu(r+h, A)}| M_f(r\lambda_{\nu(r+h, A)}) \frac{M_f((r+h)\lambda_{\nu(r+h, A)})}{M_f(r\lambda_{\nu(r+h, A)})} \\ &\leq \mu(r, A) \exp\{\ln M_f((r+h)\lambda_{\nu(r+h, A)}) - \ln M_f(r\lambda_{\nu(r+h, A)})\} \\ &= \mu(r, A) \exp \left\{ \int_{r\lambda_{\nu(r+h, A)}}^{(r+h)\lambda_{\nu(r+h, A)}} \Gamma_f(t) d \ln t \right\} \\ &\leq \mu(r, A) \exp \left\{ \Gamma_f((r+h)\lambda_{\nu(r+h, A)}) \ln(1+h/r) \right\}, \end{aligned}$$

i. e.

$$\ln \mu(r+h, A) - \ln \mu(r, A) \leq \Gamma_f((r+h)\lambda_{\nu(r+h, A)}) \ln(1+h/r). \quad (10)$$

Similarly,

$$\begin{aligned}\mu(r, A) &= |a_{\nu(r, A)}| M_f((r+h)\lambda_{\nu(r, A)}) \frac{M_f(r\lambda_{\nu(r, A)})}{M_f((r+h)\lambda_{\nu(r, A)})} \\ &\leq \mu(r+h, A) \exp \left\{ - \int_{r\lambda_{\nu(r, A)}}^{(r+h)\lambda_{\nu(r, A)}} \Gamma_f(t) d \ln t \right\} \\ &\leq \mu(r+h, A) \exp \{ -\Gamma_f(r\lambda_{\nu(r, A)}) \ln(1+h/r) \},\end{aligned}$$

i. e.

$$\ln \mu(r+h, A) - \ln \mu(r, A) \geq \Gamma_f(r\lambda_{\nu(r, A)}) \ln(1+h/r). \quad (11)$$

From (10) and (11) we obtain

$$\Gamma_f(r\lambda_{\nu(r, A)}) \frac{\ln(1+h/r)}{h} \leq \frac{\ln \mu(r+h, A) - \ln \mu(r, A)}{h} \leq \Gamma_f((r+h)\lambda_{\nu(r+h, A)}) \frac{\ln(1+h/r)}{h}.$$

Hence it follows that the functions $\ln \mu(r, A)$, $\lambda_{\nu(r, A)}$ and $\nu(r, A)$ are non-decreasing. Our reasoning is also correct if $h < 0$. Therefore, if (r_1, r_2) is an interval of constancy of the function $\nu(r, A)$ then at $h \rightarrow 0$ we obtain

$$\frac{d \ln \mu(r, A)}{dr} = \frac{\Gamma_f(r\lambda_{\nu(r, A)})}{r}, \quad r \in (r_1, r_2).$$

Since the function $\Gamma_f(r\lambda_{\nu(r, A)})$ has a finite number of discontinuities on each finite interval, we obtain the equality (9). $\square$

3 GROWTH OF $\mathfrak{M}(r, A)$, $\mu(r, A)$ AND $\lambda_{\nu(r, A)}$

At first we prove the following theorem.

Theorem 2. *If $\alpha(\ln x) \in L_{si}$, $\beta(\ln x) \in L_{si}$ and $\overline{\lim}_{n \rightarrow \infty} \frac{\ln n}{\Gamma_f(\lambda_n)} < +\infty$ then*

$$\varrho_{\alpha, \beta}[A] = \varrho_{\alpha, \beta}[\mu] := \overline{\lim}_{r \rightarrow +\infty} \frac{\alpha(\ln \mu(r, A))}{\beta(\ln r)}.$$

If $\alpha(e^x) \in L_{si}$, $\beta(\ln x) \in L_{si}$ and $\alpha(x) = o(\beta(x))$ as $x \rightarrow +\infty$ then

$$\varrho_{\alpha, \beta}[\mu] = \overline{\lim}_{r \rightarrow +\infty} \frac{\alpha(\Gamma_f(r\lambda_{\nu(r, A)}))}{\beta(\ln r)}.$$

Proof. From (3) it follows that for $q > 1$ and $r \geq 1$

$$\begin{aligned}\mu(r, A) &\leq \mathfrak{M}(r, A) \leq \sum_{n=1}^{\infty} |a_n| M_f(qr\lambda_n) \frac{M_f(r\lambda_n)}{M_f(qr\lambda_n)} \leq \mu(qr, A) \sum_{n=1}^{\infty} \exp \left\{ - \int_{r\lambda_n}^{qr\lambda_n} \Gamma_f(t) d \ln t \right\} \\ &\leq \mu(qr, A) \sum_{n=1}^{\infty} \exp \{ -\Gamma_f(r\lambda_n) \ln q \} \leq \mu(qr, A) \sum_{n=1}^{\infty} \exp \{ -\Gamma_f(\lambda_n) \ln q \}\end{aligned}$$

Since, there exist $K > 0$ such that $\ln n \leq K\Gamma_f(\lambda_n)$ for all $n \geq n_0(K)$ for $q = e^{K+1}$ we have $\Gamma_f(\lambda_n) \ln q \geq \frac{K+1}{K} \ln n$ and, thus, $\sum_{n=1}^{\infty} \exp\{-\Gamma_f(\lambda_n) \ln q\} \leq C = \text{const} < +\infty$. Therefore, $\ln \mu(r, A) \leq \ln \mathfrak{M}(r, A) \leq \ln \mu(qr, A) + \ln C$ and in view of conditions $\alpha(\ln x) \in L_{si}$ and $\beta(\ln x) \in L_{si}$ we get the equality $\varrho_{\alpha, \beta}[A] = \varrho_{\alpha, \beta}[\mu]$. The first part of Theorem 2 is proved.

Equality (9) implies

$$\ln \mu(er, A) = \ln \mu(r) + \int_r^{er} \frac{\Gamma_f(t\lambda_{\nu(t, A)})}{t} dt \geq \Gamma_f(r\lambda_{\nu(r, A)})$$

and

$$\ln \mu(r, A) = \ln \mu(1, A) + \int_1^r \frac{\Gamma_f(t\lambda_{\nu(t, A)})}{t} dt \leq \ln \mu(1, A) + \Gamma_f(r\lambda_{\nu(r, A)}) \ln r.$$

Therefore,

$$\frac{\ln \mu(r, A)}{\ln r} + o(1) \leq \Gamma_f(r\lambda_{\nu(r, A)}) \leq \ln \mu(er, A), \quad r \rightarrow +\infty. \quad (12)$$

Since $\beta(\ln x) \in L_{si}$, hence we obtain

$$\overline{\lim}_{r \rightarrow +\infty} \frac{\alpha(\Gamma_f(r\lambda_{\nu(r, A)}))}{\beta(\ln r)} \leq \overline{\lim}_{r \rightarrow +\infty} \frac{\alpha(\ln \mu(er, A))}{\beta(\ln r)} = \varrho_{\alpha, \beta}[\mu].$$

On the other hand, in view of the conditions $\alpha(e^x) \in L_{si}$ and $\alpha(x) = o(\beta(x))$ as $x \rightarrow +\infty$ we have

$$\begin{aligned} (1 + o(1))\alpha(\ln \mu(r, A)) &\leq \alpha(\Gamma_f(r\lambda_{\nu(r, A)}) \ln r) = \alpha(\exp\{\ln \Gamma_f(r\lambda_{\nu(r, A)}) + \ln \ln r\}) \\ &\leq \alpha(\exp\{2 \max\{\ln \Gamma_f(r\lambda_{\nu(r, A)}), \ln \ln r\}\}) \\ &= (1 + o(1))\alpha(\exp\{\max\{\ln \Gamma_f(r\lambda_{\nu(r, A)}), \ln \ln r\}\}) \\ &= (1 + o(1)) \max\{\alpha(\Gamma_f(r\lambda_{\nu(r, A)})), \alpha(\ln r)\} \leq (1 + o(1))(\alpha(\Gamma_f(r\lambda_{\nu(r, A)})) + \alpha(\ln r)) \\ &= (1 + o(1))(\alpha(\Gamma_f(r\lambda_{\nu(r, A)})) + o(\beta(\ln r))), \quad r \rightarrow +\infty, \end{aligned}$$

i. e. $\varrho_{\alpha, \beta}[\mu] \leq \overline{\lim}_{r \rightarrow +\infty} \frac{\alpha(\Gamma_f(r\lambda_{\nu(r, A)}))}{\beta(\ln r)}$. The proof of Theorem 2 is complete. $\square$

Remark that if all $f_k \geq 0$ then $M_f(r) = f(r)$. In this case the following theorem is true.

Theorem 3. Let $\alpha \in L_{si}$, $\beta(\ln x) \in L_{si}$ and for all $r \geq r_0$

$$\ln m < h \leq \frac{d \ln \ln f(r)}{d \ln r} \leq H < +\infty. \quad (13)$$

If series (5) regularly converge in $\mathbb{C}$ and $A(z) = (A_1 * \dots * A_p)_m(z)$ then $\varrho_{\alpha, \beta}[A] \leq \max\{\varrho_{\alpha, \beta}[A_j] : 1 \leq j \leq p\}$.

Moreover, if the function A_1 is dominant then $\varrho_{\alpha, \beta}[A_j] \leq \varrho_{\alpha, \beta}[A_1] = \varrho_{\alpha, \beta}[A]$.

Proof. From (13) it follows that for $\eta = \frac{\ln m}{H} + 1$ and for some $\xi \in (r/\eta, r)$

$$\begin{aligned} \ln \ln M_f(r) - \ln \ln M_f(r/\eta) &= \ln \ln f(r) - \ln \ln f(r/\eta) = (\ln \ln f(\xi))'(1 - 1/\eta)r \\ &\leq (\ln \ln f(\xi))'(1 - 1/\eta)\eta\xi = (\eta - 1)\frac{d \ln \ln f(\xi)}{d \ln \xi} \leq (\eta - 1)H \leq \ln m, \end{aligned}$$

i. e. $M_f(r) \leq M_f(r/\eta)^m$. Therefore, (8) implies

$$|a_n| M_f(r\lambda_n) \leq \sum_{k_1 + \dots + k_p = m} |c_{k_1 \dots k_p}| (|a_{n,1}| M_f(r\lambda_n/\eta))^{k_1} \cdot \dots \cdot (|a_{n,p}| M_f(r\lambda_n/\eta))^{k_p},$$

i. e.

$$\mu(r, A) \leq \sum_{k_1 + \dots + k_p = m} |c_{k_1 \dots k_p}| \mu(r/\eta, A_1)^{k_1} \cdot \dots \cdot \mu(r/\eta, A_p)^{k_p}.$$

Since $\ln^+(c_1 + \dots + c_n) \leq \ln^+ c_1 + \dots + \ln^+ c_n + \ln n$, hence for all r enough large we get

$$\begin{aligned} \ln \mu(r, A) &\leq \sum_{k_1 + \dots + k_p = m} \ln (|c_{k_1 \dots k_p}| \mu(r/\eta, A_1)^{k_1} \cdot \dots \cdot \mu(r/\eta, A_p)^{k_p}) + C_1 \\ &\leq \sum_{k_1 + \dots + k_p = m} (k_1 \ln \mu(r/\eta, A_1) + \dots + k_p \ln \mu(r/\eta, A_p)) + C_2, \end{aligned}$$

where C_1 and C_2 are const > 0 . But $\ln \mu(r, A_j) \leq \alpha^{-1}(\varrho_{\alpha, \beta}[A_j] + \varepsilon)\beta(\ln r) \leq \alpha^{-1}(\varrho + \varepsilon)\beta(\ln r)$ for all $\varepsilon > 0$ and all $r \geq r_0(\varepsilon)$, where $\varrho = \max\{\varrho_{\alpha, \beta}[A_j] : 1 \leq j \leq p\}$. Therefore, $\ln \mu(r, A) \leq m\alpha^{-1}(\varrho + \varepsilon)\beta(\ln r - 1) + C_2$, whence in view of the conditions $\alpha \in L_{si}$, $\beta(\ln x) \in L_{si}$ and of the arbitrariness of ε we obtain $\varrho_{\alpha, \beta}[A] \leq \varrho$. The first part of Theorem 3 is proved.

If the function A_1 is dominant then, as above, we have $|a_n| \geq c|a_{n,1}|^m$ for $n \geq n_0^*$, where $c = \text{const} > 0$. From (13) it follows that for $\zeta = 1 - \frac{\ln m}{h}$ and for some $\xi \in (\zeta r, r)$, as above, we have

$$\ln \ln M_f(r) - \ln \ln M_f(\zeta r) = (\ln \ln f(\xi))'(1 - \zeta)r \geq (1 - \zeta)\frac{d \ln \ln f(\xi)}{d \ln \xi} \geq (1 - \zeta)h \geq \ln m,$$

i. e. $M_f(r) \geq M_f(r\zeta)^m$. Therefore, $\mu(r, A) \geq c\mu(r\zeta, A)^m$, whence the inequality $\varrho_{\alpha, \beta}[A_j] \leq \varrho_{\alpha, \beta}[A_1] \leq \varrho_{\alpha, \beta}[A]$ follows and $\varrho_{\alpha, \beta}[A] = \varrho_{\alpha, \beta}[A_1]$. The proof of Theorem 3 is complete. $\square$

4 GROWTH OF THE DERIVATIVE

Clearly, $A^{(k)}(z) = \sum_{n=1}^{\infty} a_n^k \lambda_n f A^{(k)}(\lambda_n z)$ and $\mathfrak{M}(r, A^{(k)}) = \sum_{n=1}^{\infty} |a_n| \lambda_n^k M_{f^{(k)}}(r\lambda_n)$. The following statement indicates that for each $k \geq 1$ the functions $A^{(k)}$ and A have the same growth.

Theorem 4. *If $\alpha \in L_{si}$, $\beta(\ln x) \in L_{si}$ and $\alpha(x) = o(\beta(x))$ as $x \rightarrow +\infty$ then $\varrho_{\alpha, \beta}[A^{(k)}] = \varrho_{\alpha, \beta}[A]$.*

Proof. It suffices to prove Theorem 3 for $k = 1$. From Cauchy formula $f'(z) = \frac{1}{2\pi i} \int_{|\tau-z|=\lambda_n} \frac{f(\tau)}{(\tau-z)^2} d\tau$ we obtain $M_{f'}(r) \leq \frac{M_f(r+\lambda_n)}{\lambda_n}$ for all r and n . Therefore, choosing $r\lambda_n$ instead r we get

$$\mathfrak{M}(r, A') \leq \sum_{n=1}^{\infty} |a_n| \lambda_n \frac{M_f(r\lambda_n + \lambda_n)}{\lambda_n} = \sum_{n=1}^{\infty} |a_n| M_f((r+1)\lambda_n) = \mathfrak{M}(r+1, A),$$

whence the inequality $\varrho_{\alpha,\beta}[A'] \leq \varrho_{\alpha,\beta}[A]$ follows.

On the other hand, $f(z) - f(0) = \int_0^z f'(z) dz$, whence $M_f(r) \leq |f(0)| + r M_{f'}(r)$ and, thus,

$$\begin{aligned} \mathfrak{M}(r, A) &\leq \sum_{n=1}^{\infty} |a_n| (|f(0)| + r\lambda_n M_{f'}(r\lambda_n)) = (1 + o(1))r \sum_{n=1}^{\infty} |a_n| \lambda_n M_{f'}(r\lambda_n) \\ &= (1 + o(1))r \mathfrak{M}(r, A'), \quad r \rightarrow +\infty, \end{aligned}$$

whence, as above,

$$\begin{aligned} \alpha(\ln \mathfrak{M}(r, A)) &\leq \alpha(\ln \mathfrak{M}(r, A)) + \ln r + o(1) \leq (1 + o(1))\alpha(2 \max\{\ln \mathfrak{M}(r, A), \ln r\}) \\ &= (1 + o(1))\alpha(\max\{\ln \mathfrak{M}(r, A), \ln r\}) \leq (1 + o(1))(\alpha(\mathfrak{M}(r, A)) + \alpha(\ln r)), \quad r \rightarrow +\infty, \end{aligned}$$

i. e. $\varrho_{\alpha,\beta}[A] \leq \varrho_{\alpha,\beta}[A']$. Theorem 4 is proved. $\square$

5 GROWTH OF THE RATIO $\frac{\mu(r, (A_1^{(k)} * \dots * A_p^{(k)})_m)}{\mu(r, ((A_1 * \dots * A_p)_m)^{(k)})}$

Let $j \geq k$. Since

$$(A_1^{(j)} * \dots * A_p^{(j)})_m(z) = \left(\sum_{k_1 + \dots + k_p = m} c_{k_1 \dots k_p} a_{n,1}^{k_1} \cdot \dots \cdot a_{n,p}^{k_p} \right) \lambda_n^{mj} f^{(j)}(\lambda_n z)$$

and

$$((A_1 * \dots * A_p)_m)^{(j)}(k) = \sum_{n=1}^{\infty} \left(\sum_{k_1 + \dots + k_p = m} c_{k_1 \dots k_p} a_{n,1}^{k_1} \cdot \dots \cdot a_{n,p}^{k_p} \right) \lambda_n^k f^{(k)}(\lambda_n z),$$

we have (with $a_n = \sum_{k_1 + \dots + k_p = m} c_{k_1 \dots k_p} a_{n,1}^{k_1} \cdot \dots \cdot a_{n,p}^{k_p}$)

$$\begin{aligned} \mu(r, (A_1^{(j)} * \dots * A_p^{(j)})_m) &= |a_{\nu(r, (A_1^{(j)} * \dots * A_p^{(j)})_m)}| \lambda_{\nu(r, (A_1^{(j)} * \dots * A_p^{(j)})_m)}^{mj} M_{f^{(j)}}(r \lambda_{\nu(r, (A_1^{(j)} * \dots * A_p^{(j)})_m)}) \\ &= |a_{\nu(r, (A_1^{(j)} * \dots * A_p^{(j)})_m)}| \lambda_{\nu(r, (A_1^{(j)} * \dots * A_p^{(j)})_m)}^k M_{f^{(k)}}(r \lambda_{\nu(r, (A_1^{(j)} * \dots * A_p^{(j)})_m)}) \times \\ &\quad \times \lambda_{\nu(r, (A_1^{(j)} * \dots * A_p^{(j)})_m)}^{mj-k} \frac{M_{f^{(j)}}(r \lambda_{\nu(r, (A_1^{(j)} * \dots * A_p^{(j)})_m)})}{M_{f^{(k)}}(r \lambda_{\nu(r, (A_1^{(j)} * \dots * A_p^{(j)})_m)})} \\ &\leq \mu(r, ((A_1 * \dots * A_p)_m)^{(k)}) \lambda_{\nu(r, (A_1^{(j)} * \dots * A_p^{(j)})_m)}^{mj-k} \frac{M_{f^{(j)}}(r \lambda_{\nu(r, (A_1^{(j)} * \dots * A_p^{(j)})_m)})}{M_{f^{(k)}}(r \lambda_{\nu(r, (A_1^{(j)} * \dots * A_p^{(j)})_m)})}. \end{aligned}$$

Similarly,

$$\begin{aligned}
\mu(r, ((A_1 * \dots * A_p)_m)^{(k)}) &= |a_{\nu(r, ((A_1 * \dots * A_p)_m)^{(k)})}| \lambda_{\nu(r, ((A_1 * \dots * A_p)_m)^{(k)})}^k M_{f^{(k)}}(r \lambda_{\nu(r, ((A_1 * \dots * A_p)_m)^{(k)})}) \\
&= |a_{\nu(r, ((A_1 * \dots * A_p)_m)^{(k)})}| \lambda_{\nu(r, ((A_1 * \dots * A_p)_m)^{(k)})}^{mj} M_{f^{(j)}}(r \lambda_{\nu(r, ((A_1 * \dots * A_p)_m)^{(k)})}) \times \\
&\quad \times \lambda_{\nu(r, ((A_1 * \dots * A_p)_m)^{(k)})}^{k-mj} \frac{M_{f^{(k)}}(r \lambda_{\nu(r, ((A_1 * \dots * A_p)_m)^{(k)})})}{M_{f^{(j)}}(r \lambda_{\nu(r, ((A_1 * \dots * A_p)_m)^{(k)})})} \\
&\leq \mu(r, (A_1^{(j)} * \dots * A_p^{(j)})_m) \lambda_{\nu(r, ((A_1 * \dots * A_p)_m)^{(k)})}^{k-mj} \frac{M_{f^{(k)}}(r \lambda_{\nu(r, ((A_1 * \dots * A_p)_m)^{(k)})})}{M_{f^{(j)}}(r \lambda_{\nu(r, ((A_1 * \dots * A_p)_m)^{(k)})})}.
\end{aligned}$$

Therefore, the following lemma is proved.

Lemma 3. *If functions (5) regularly converge in $\mathbb{C}$ and $j \geq k$ then*

$$\begin{aligned}
\lambda_{\nu(r, ((A_1 * \dots * A_p)_m)^{(k)})}^{mj-k} \frac{M_{f^{(j)}}(r \lambda_{\nu(r, ((A_1 * \dots * A_p)_m)^{(k)})})}{M_{f^{(k)}}(r \lambda_{\nu(r, ((A_1 * \dots * A_p)_m)^{(k)})})} &\leq \frac{\mu(r, (A_1^{(j)} * \dots * A_p^{(j)})_m)}{\mu(r, ((A_1 * \dots * A_p)_m)^{(k)})} \\
&\leq \lambda_{\nu(r, (A_1^{(j)} * \dots * A_p^{(j)})_m)}^{mj-k} \frac{M_{f^{(j)}}(r \lambda_{\nu(r, (A_1^{(j)} * \dots * A_p^{(j)})_m)})}{M_{f^{(k)}}(r \lambda_{\nu(r, (A_1^{(j)} * \dots * A_p^{(j)})_m)})}.
\end{aligned} \tag{14}$$

If we put

$$\mathfrak{G}(r, A, k) := \sqrt[m-1]{\frac{r^{(m-1)k} \mu(r, (A_1^{(k)} * \dots * A_p^{(k)})_m)}{\mu(r, ((A_1 * \dots * A_p)_m)^{(k)})}}$$

then the following theorem is true.

Theorem 5. *Let $\alpha(e^x) \in L_{si}$, $\beta(\ln x) \in L_{si}$, $\alpha(x) = o(\beta(x))$ as $x \rightarrow +\infty$ and $\overline{\lim}_{n \rightarrow \infty} \frac{\ln n}{\Gamma_f(\lambda_n)} < +\infty$. Suppose that all $f_k \geq 0$ and $\ln m < h \leq \frac{d \ln \ln f(r)}{d \ln r} \leq H < +\infty$ for all $r \geq r_0$. If series (5) regularly converge in $\mathbb{C}$, $A(z) = (A_1 * \dots * A_p)_m(z)$ and the function A_1 is dominant then*

$$\overline{\lim}_{r \rightarrow +\infty} \frac{\alpha(\Gamma_f(\mathfrak{G}(r, A, k)))}{\beta(\ln r)} = \varrho_{\alpha, \beta}[A]. \tag{15}$$

Proof. If $j = k$ from (14) we get $r \lambda_{\nu(r, ((A_1 * \dots * A_p)_m)^{(k)})} \leq \mathfrak{G}(r, A, k) \leq r \lambda_{\nu(r, (A_1^{(k)} * \dots * A_p^{(k)})_m)}$, whence

$$\Gamma_f(r \lambda_{\nu(r, ((A_1 * \dots * A_p)_m)^{(k)})}) \leq \Gamma_f(\mathfrak{G}(r, A, k)) \leq \Gamma_f(r \lambda_{\nu(r, (A_1^{(k)} * \dots * A_p^{(k)})_m)})$$

and, thus, by Theorem 2

$$\varrho_{\alpha, \beta}[\mu(\cdot, ((A_1 * \dots * A_p)_m)^{(k)})] \leq \overline{\lim}_{r \rightarrow +\infty} \frac{\alpha(\Gamma_f(\mathfrak{G}(r, A, k)))}{\beta(\ln r)} \leq \varrho_{\alpha, \beta}[\mu(\cdot, (A_1^{(k)} * \dots * A_p^{(k)})_m)]. \tag{16}$$

By Theorems 2 and 4

$$\varrho_{\alpha, \beta}[\mu(\cdot, ((A_1 * \dots * A_p)_m)^{(k)})] = \varrho_{\alpha, \beta}[(A_1 * \dots * A_p)_m^{(k)}] = \varrho_{\alpha, \beta}[(A_1 * \dots * A_p)_m] = \varrho_{\alpha, \beta}[A]$$

and by Theorems 2, 3 and 4

$$\begin{aligned}
\varrho_{\alpha, \beta}[\mu(\cdot, (A_1^{(k)} * \dots * A_p^{(k)})_m)] &= \varrho_{\alpha, \beta}[(A_1^{(k)} * \dots * A_p^{(k)})_m] \leq \max\{\varrho_{\alpha, \beta}[A_j^{(k)}] : 1 \leq j \leq p\} \\
&= \max\{\varrho_{\alpha, \beta}[A_j] : 1 \leq j \leq p\} = \varrho_{\alpha, \beta}[A_1] = \varrho_{\alpha, \beta}[A].
\end{aligned}$$

Therefore, (16) implies (15). Theorem 5 is proved. $\square$

As an example, consider series in the system of Mittag-Leffler functions. Let $0 < \varrho < +\infty$ and $E_\varrho(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+k/\varrho)}$ be the Mittag-Leffler function. Denote $A_\varrho(z) = \sum_{n=1}^{\infty} a_n E_\varrho(\lambda_n z)$ and $A_{\varrho,j}(z) = \sum_{n=1}^{\infty} a_{n,j} E_\varrho(\lambda_n z)$, $1 \leq j \leq p$. It is well known [2, p. 115] that $M_{E_\varrho}(r) = E_\varrho(r) = (1+o(1))\varrho e^{r^\varrho}$ as $r \rightarrow +\infty$ and this equality can be differentiated. Therefore, $\Gamma_{E_\varrho}(r) = (1+o(1))\varrho r^\varrho$ and $\frac{d \ln \ln E_\varrho(r)}{d \ln r} = (1+o(1))\varrho$ as $r \rightarrow +\infty$. Therefore, for $\alpha(x) = \ln^+ \ln x$ and $\beta(x) = x^+$ we obtain the following statement.

Corollary 1. *Let $\ln n = O(\lambda_n^\varrho)$ as $n \rightarrow \infty$ and the series $\sum_{n=1}^{\infty} a_{n,j} E_\varrho(\lambda_n z)$ regularly converge in $\mathbb{C}$. If $\ln m < \varrho$, $A_\varrho(z) = (A_{\varrho,1} * \dots * A_{\varrho,p})_m(z)$ and the function $A_{\varrho,1}$ is dominant then*

$$\overline{\lim}_{r \rightarrow +\infty} \frac{1}{\ln r} \ln \ln \frac{\mu(r, (A_{\varrho,1}^{(k)} * \dots * A_{\varrho,p}^{(k)})_m)}{\mu(r, ((A_{\varrho,1} * \dots * A_{\varrho,p})_m)^{(k)})} = \overline{\lim}_{r \rightarrow +\infty} \frac{\ln \ln \ln \mathfrak{M}(r, A_\varrho)}{\ln r}$$

We remark that the condition $\alpha(e^x) \in L_{si}$ used in the proof of equality $\varrho_{\alpha,\beta}[\mu] = \overline{\lim}_{r \rightarrow +\infty} \frac{\alpha(\Gamma_f(r\lambda_{\nu(r,A)}))}{\beta(\ln r)}$. But in the case when $\alpha(x) = \ln^+ x$ and $\beta(x) = x^+$ from (12) we get $\ln \ln \mu(r, A) \leq \ln(\Gamma_f(r\lambda_{\nu(r,A)}) + o(1)) + \ln \ln r$ and, thus,

$$\overline{\lim}_{r \rightarrow +\infty} \frac{\ln \ln \mu(r, A)}{\ln r} = \overline{\lim}_{r \rightarrow +\infty} \frac{\ln \Gamma_f(r\lambda_{\nu(r,A)})}{\ln r}.$$

As a result, we arrive at the following statement.

Proposition 1. *Let $\ln n = O(\lambda_n^\varrho)$ as $n \rightarrow \infty$ and the series $\sum_{n=1}^{\infty} a_{n,j} E_\varrho(\lambda_n z)$ regularly converge in $\mathbb{C}$. If $\ln m < \varrho$, $A_\varrho(z) = (A_{\varrho,1} * \dots * A_{\varrho,p})_m(z)$ and the function $A_{\varrho,1}$ is dominant then*

$$\varrho \overline{\lim}_{r \rightarrow +\infty} \frac{\ln \mathfrak{G}(r, A_\varrho, k)}{\ln r} = \overline{\lim}_{r \rightarrow +\infty} \frac{\ln \ln \mathfrak{M}(r, A_\varrho)}{\ln r}.$$

Put $\sigma = r^\varrho$, $l_n = \lambda_n^\varrho$ and $D(\sigma) = \sum_{n=1}^{\infty} |a_n| e^{l_n \sigma}$. Then $\mu(r, A_\varrho) = (1+o(1))\mu(\sigma, D)$ as $\sigma \rightarrow +\infty$, where $\mu(\sigma, D)$ is maximal term of entire Dirichlet series D . It is known [8, p. 26] that if $\ln n = O(l_n)$ as $n \rightarrow \infty$, $\alpha_1 \in L_{si}$, $\beta_1(\ln x) \in L_{si}$ and $\frac{d\beta_1^{-1}(c\alpha_1(x))}{d \ln x} = O(1)$ as $n \rightarrow \infty$ for each $c \in (0, +\infty)$ then

$$\overline{\lim}_{\sigma \rightarrow +\infty} \frac{\alpha_1(\ln D(\sigma))}{\beta_1(\sigma)} = \overline{\lim}_{\sigma \rightarrow +\infty} \frac{\alpha_1(\ln \mu(\sigma, D))}{\beta_1(\sigma)} = \overline{\lim}_{n \rightarrow \infty} \frac{\alpha_1(l_n)}{\beta_1\left(\frac{1}{l_n} \ln \frac{1}{|a_n|}\right)}.$$

If we put $\alpha_1(x) = \alpha(x)$, $\beta_1(x) = \beta((\ln x)/\varrho)$ then here, under conditions $\alpha(e^x) \in L_{si}$, $\beta(\ln x) \in L_{si}$ and $\frac{d \exp\{\varrho \beta^{-1}(c\alpha_1(x))\}}{d \ln x} = O(1)$ as $n \rightarrow \infty$ for each $c \in (0, +\infty)$, we get

$$\begin{aligned} \varrho_{\alpha,\beta}[A_\varrho] &= \overline{\lim}_{r \rightarrow +\infty} \frac{\ln \mathfrak{M}(r, A_\varrho)}{\beta(\ln r)} = \overline{\lim}_{r \rightarrow +\infty} \frac{\alpha(\ln D(\sigma))}{\beta((\ln \sigma)\varrho)} \\ &= \overline{\lim}_{n \rightarrow \infty} \frac{\alpha(\lambda_n^\varrho)}{\beta\left(\frac{1}{\varrho} \ln \left(\frac{1}{\lambda_n^\varrho} \ln \frac{1}{|a_n|}\right)\right)} = \overline{\lim}_{n \rightarrow \infty} \frac{\alpha(\lambda_n)}{\beta\left(\frac{1}{\varrho} \ln \left(\frac{1}{\lambda_n^\varrho} \ln \frac{1}{|a_n|}\right)\right)}, \end{aligned} \quad (17)$$

whence for $\alpha(x) = (\ln^+ \ln x)^q$, $0 < q < 1$, and $\beta(x) = x^+$ we obtain

$$\overline{\lim}_{r \rightarrow +\infty} \frac{\ln^q \ln \ln \mathfrak{M}(r, A_\varrho)}{\ln r} = \overline{\lim}_{n \rightarrow \infty} \frac{\varrho \ln^q \ln \lambda_n}{\ln \left(\frac{1}{\lambda_n^\varrho} \ln \frac{1}{|a_n|} \right)},$$

6 DISCUSSION OPEN PROBLEMS

We were unable to solve the following actual problems.

1. Using inequalities (14) prove an analogue of Theorem 5 for $j > k$.
2. In the general case, find a formula for finding the generalized order $\varrho_{\alpha, \beta}[A] = \overline{\lim}_{r \rightarrow +\infty} \frac{\alpha(\ln \mathfrak{M}(r, A))}{\beta(\ln r)}$ in terms of coefficients and prove an analogue of equality (17).
3. Is it possible to establish a connection between the growth of functions A_j and function A if there is no dominant function among the functions A_j ?

REFERENCES

- [1] Bieberbach L. *Analytische Fortsetzung*. Springer-Verlag, Berlin, 1955. doi:10.1002/zamm.19550350918.
- [2] Gol'dberg A.A., Ostrovskiy I.V. Distribution of values of meromorphic functions. Nauka, Moscow, 1976. (in Russian)
- [3] Hadamard J. *La serie de Taylor et son prolongement analytique*. Scientia phys.- math. 1901, **12**, 43-62.
- [4] Hadamard J. *Theoreme sur le series entieres*. Acta math. 1899, **22**, 55-63.
- [5] Leont'ev A.F. Generalizations of exponential series. Nauka, Moscow, 1981. (in Russian)
- [6] Mulyava O.M., Sheremeta M.M. *Compositions of Dirichlet series similar to the Hadamard compositions, and convergence classes*. Mat. Stud. 2019. **51** (1), 25-34. doi:10.15330/ms.51.1.25-34.
- [7] Sheremeta M.N. *Connection between the growth of the maximum of the modulus of an entire function and the moduli of the coefficients of its power series expansion*. Izv. Vyssh. Uchebn. Zaved. Mat. 1967, no. 2, 100-108. (in Russian)
- [8] Sheremeta M.M. Entire Dirichlet series. ICDO, Kyiv, 1993. (in Ukrainian).
- [9] Sheremeta M.M. *Hadamard compositions of Gelfond-Leont'ev derivatives*. Axioms. 2022. **11**, 478. doi.org/10.3390/axioms1090478.
- [10] Sheremeta M.M. *On the growth of series in systems of functions and Laplace-Stieltjes integrals*. Mat. Stud. 2021. **55**(2), 124-131.
- [11] Sheremeta M.M. *Relative growth of series in system functions and Laplace-Stieltjes type integrals*. Axioms. 2021. - **10**, 43.
- [12] Vynnytskyi B.V. Some approximation properties of generalized systems of exponentials. Drohobych, 1991. Dep. in UkrNIINTI 25.02.1991. (in Russian)

Received 02.01.2023

Шеремета М.М. *Композиції Адамара рядів за системою функцій* // Буковинський матем. журнал — 2023. — Т.11, №1. — С. 39–51.

Для регулярно збіжних в $\mathbb{C}$ рядів $A_j(z) = \sum_{n=1}^{\infty} a_{n,j} f(\lambda_n z)$, $1 \leq j \leq p$, де f - ціла трансцендентна функція, досліджується асимптотичне поведіння адамарової композиції $A(z) = (A_1 * \dots * A_p)_m(z) = \sum_{n=1}^{\infty} \left(\sum_{k_1 + \dots + k_p = m} c_{k_1 \dots k_p} a_{n,1}^{k_1} \cdot \dots \cdot a_{n,p}^{k_p} \right) f(\lambda_n z)$ роду m . Функція A_1 називається домінантною, якщо $|c_{m0\dots 0}| |a_{n,1}|^m \neq 0$ і $|a_{n,j}| = o(|a_{n,1}|)$ при $n \rightarrow \infty$ для $2 \leq j \leq p$. Узагальненим порядком функції A_j називається величина $\varrho_{\alpha,\beta}[A_j] = \overline{\lim}_{r \rightarrow +\infty} \frac{\alpha(\ln \mathfrak{M}(r, A_j))}{\beta(\ln r)}$, де $\mathfrak{M}(r, A_j) = \sum_{n=1}^{\infty} |a_{n,j}| M_f(r \lambda_n)$, $M_f(r) = \max\{|f(z)| : |z| = r\}$, а функції α і β є додатні, неперервні і зростаючі до $+\infty$.

За певних умов на α , β , $M_f(r)$ і (λ_n) доведено, що якщо серед функцій A_j існує домінантна, то $\varrho_{\alpha,\beta}[A] = \max\{\varrho_{\alpha,\beta}[A_j] : 1 \leq j \leq p\}$. У термінах узагальнених порядків встановлено зв'язок між ростом максимальних членів функцій $(A_1^{(k)} * \dots * A_p^{(k)})_m$ і $((A_1 * \dots * A_p)_m)^{(k)}$. Сформульовано нерозв'язані проблеми.