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ON THE GROWTH OF THE MAXIMUM MODULUS OF DIRICHLET SERIES

For an entire Dirichlet series $F(s) = \sum_{n=0}^{\infty} f_n \exp\{s\lambda_n\}$ with $0 \leq \lambda_n \uparrow +\infty$, a connection between the growth of the maximum modulus $M(\sigma, F) = \sup\{|F(\sigma + it)| : t \in \mathbb{R}\}$ and the decrease of the coefficients is studied. For example, it is proved that if $\overline{\lim}_{k \rightarrow \infty} \alpha(\lambda_k)/\beta\left(\frac{1}{\lambda_k} \ln \frac{1}{|f_k|}\right) = Q > 0$, where α, β are positive continuous functions on $[x_0, +\infty)$ increasing to $+\infty$, then $\overline{\lim}_{\sigma \rightarrow +\infty} (\exp\{\alpha(\ln M(\beta^{-1}(\beta(\sigma) + \ln q), F))\} - p \exp\{\alpha(\ln M(\sigma, F))\}) = +\infty$ for any $q > 1$ and $p > 1$ such that $\ln p / \ln q < Q$. Similar results are obtained for Dirichlet series with zero abscissa of absolute convergence.

Key words and phrases: Dirichlet series, maximum modulus, generalized order.

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INTRODUCTION

For an entire transcendental function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ let $M_f(r) = \max\{|f(z)| : |z| = r\}$. It is known [5] that $M_f(ar)/M_f(r) \nearrow +\infty$ as $r \rightarrow +\infty$ for every $a > 1$. S. Singh [9] gave a more simple (on his opinion) proof of this fact, using the relation between $M_f(r)$ and the maximal term of the power expansion of f . A much simpler proof is given in [6] for entire functions given by both power series and Dirichlet series.

Let's remark that Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} f_n \exp\{s\lambda_n\}, \quad s = \sigma + it, \quad (1)$$

with non-negative increasing to $+\infty$ exponents λ_n are direct generalization of power series.

We suppose that series (1) has the abscissa of absolute convergence $\sigma_a \in (-\infty, +\infty]$, and for $\sigma < \sigma_a$ we put $M(\sigma, F) = \sup\{|F(\sigma + it)| : t \in \mathbb{R}\}$. If $\sigma_a = +\infty$ and series (1) is not reduced to an exponential polynomial then [6] $M(\sigma + h, F)/M(\sigma, F) \nearrow +\infty$ as $\sigma \rightarrow +\infty$ for every $h > 0$.

УДК 517.537.72

2010 *Mathematics Subject Classification:* 30B50.

Information on some grant ...

By L we denote a class of continuous non-negative on $(-\infty, +\infty)$ functions α such that $\alpha(x) = \alpha(x_0) \geq 0$ for $x \leq x_0$ and $\alpha(x) \uparrow +\infty$ as $x_0 \leq x \rightarrow +\infty$. We say that $\alpha \in L^0$, if $\alpha \in L$ and $\alpha((1+o(1))x) = (1+o(1))\alpha(x)$ as $x \rightarrow +\infty$. Finally, $\alpha \in L_{si}$, if $\alpha \in L$ and $\alpha(cx) = (1+o(1))\alpha(x)$ as $x \rightarrow +\infty$ for each $c \in (0, +\infty)$, i. e. α is a slowly increasing function. Clearly, $L_{si} \subset L^0$. We remark that for every $\alpha \in L_{si}$ there exists $\alpha_1 \in L$ such that $\alpha_1(x) = (1+o(1))\alpha(x)$ and $\frac{x\alpha'_1(x)}{\alpha_1(x)} \rightarrow 0$ as $x \rightarrow +\infty$. Therefore, in the future we will assume that the function $\alpha \in L_{si}$ satisfies the condition $\frac{x\alpha'(x)}{\alpha(x)} \rightarrow 0$ as $x \rightarrow +\infty$.

In [6] the following theorems are proved.

Theorem A. Let $\sigma_a = +\infty$, $\gamma \in L$, $p > 1$. If $\overline{\lim}_{n \rightarrow \infty} \frac{\ln \ln \gamma(\lambda_n)}{\ln \left(\frac{1}{\lambda_n} \ln \frac{1}{|f_n|} \right)} = \eta^* > 0$ then

$$\overline{\lim}_{\sigma \rightarrow +\infty} \frac{\gamma(\ln M(q\sigma, F))}{\gamma^p(\ln M(\sigma, F))} = +\infty \quad (2)$$

for each $q > p^{1/\eta^*}$.

If the function γ is continuously differentiable, $\ln \gamma(e^x) \in L_{si}$, $\ln \ln \gamma(\lambda_{n+1}) = (1+o(1)) \ln \ln \gamma(\lambda_n)$, $\ln n = O(\lambda_n)$ as $n \rightarrow \infty$, $\kappa_n[F] := \frac{\ln |f_n| - \ln |f_{n+1}|}{\lambda_{n+1} - \lambda_n} \nearrow +\infty$

as $n_0 \leq n \rightarrow \infty$ and $\underline{\lim}_{n \rightarrow \infty} \frac{\ln \ln \gamma(\lambda_n)}{\ln \left(\frac{1}{\lambda_n} \ln \frac{1}{|f_n|} \right)} = \eta_* < +\infty$, then

$$\underline{\lim}_{\sigma \rightarrow +\infty} \frac{\gamma(\ln M(q\sigma, F))}{\gamma^p(\ln M(\sigma, F))} = 0 \quad (3)$$

for each $q < p^{1/\eta_*}$.

Theorem B. Let $\sigma_a = 0$, $p > 1$, $\gamma(e^x) \in L^0$ and $\ln \ln \gamma(x) = o(\ln x)$ as $x \rightarrow +\infty$. If $\overline{\lim}_{n \rightarrow \infty} \frac{\ln \ln \gamma(\lambda_n)}{\ln(\lambda_n / \ln^+ |f_n|)} = \eta^* > 0$ then

$$\overline{\lim}_{\sigma \uparrow 0} \frac{\gamma(\ln M(\sigma/q, F))}{\gamma^p(\ln M(\sigma, F))} = +\infty$$

for each $q > p^{1/\eta^*}$.

If the function γ is continuously differentiable, $\ln \gamma(x) \in L_{si}$, $\kappa_n[F] \nearrow 0$ and $\ln \ln \gamma(\lambda_{n+1}) = (1+o(1)) \ln \ln \gamma(\lambda_n)$ as $n_0 \leq n \rightarrow \infty$, $\overline{\lim}_{n \rightarrow \infty} \frac{\ln \ln n}{\ln \lambda_n} < 1$ and $\underline{\lim}_{n \rightarrow \infty} \frac{\ln \ln \gamma(\lambda_n)}{\ln(\lambda_n / \ln^+ |f_n|)} = \eta_* < +\infty$ then

$$\underline{\lim}_{\sigma \uparrow 0} \frac{\gamma(\ln M(\sigma/q, F))}{\gamma^p(\ln M(\sigma, F))} = 0$$

for each $q < p^{1/\eta_*}$.

For the proof of these theorems in [6] the following theorem was used.

Theorem C. Let $q > 1$, $p > 1$, $\gamma \in L$ and Φ be a positive function continuous on $(x_0, +\infty)$ and increasing to $+\infty$. If $\overline{\lim}_{\sigma \rightarrow +\infty} \frac{\gamma(\Phi(q\sigma))}{\gamma^p(\Phi(\sigma))} \leq \xi_1 < +\infty$ then $\overline{\lim}_{\sigma \rightarrow +\infty} \frac{\ln \ln \gamma(\Phi(\sigma))}{\ln \sigma} \leq \frac{\ln p}{\ln q}$ and if $\underline{\lim}_{\sigma \rightarrow +\infty} \frac{\gamma(\Phi(q\sigma))}{\gamma^p(\Phi(\sigma))} \geq \xi_2 > 0$ then $\underline{\lim}_{\sigma \rightarrow +\infty} \frac{\ln \ln \gamma(\Phi(\sigma))}{\ln \sigma} \geq \frac{\ln p}{\ln q}$.

Using the generalized growth scale, here we will continue the studies started in [6].

1 ENTIRE DIRICHLET SERIES

If $\alpha \in L$, $\beta \in L$ and F is an entire function then the quantities

$$\varrho_{\alpha,\beta}[F] := \overline{\lim}_{\sigma \rightarrow +\infty} \frac{\alpha(\ln M(\sigma, F))}{\beta(\sigma)}, \quad \lambda_{\alpha,\beta}[F] := \underline{\lim}_{\sigma \rightarrow +\infty} \frac{\alpha(\ln M(\sigma, F))}{\beta(\sigma)}$$

are called [7], [8] the generalized (α, β) -order and the generalized lower (α, β) -order of F respectively. We put

$$Q_{\alpha,\beta}[F] = \overline{\lim}_{k \rightarrow \infty} \frac{\alpha(\lambda_k)}{\beta\left(\frac{1}{\lambda_k} \ln \frac{1}{|f_k|}\right)}, \quad q_{\alpha,\beta}[F] = \underline{\lim}_{k \rightarrow \infty} \frac{\alpha(\lambda_k)}{\beta\left(\frac{1}{\lambda_k} \ln \frac{1}{|f_k|}\right)}.$$

Then the following lemma is true [7], [8].

Lemma 1. *Let $\alpha \in L_{si}$, $\beta \in L^0$ and $\frac{d\beta^{-1}(c\alpha(x))}{d \ln x} = O(1)$ as $x \rightarrow +\infty$ for each $c \in (0, +\infty)$. If $\ln k = o(\lambda_k \beta^{-1}(c\alpha(\lambda_k)))$ as $k \rightarrow \infty$ for each $c \in (0, +\infty)$ then $\varrho_{\alpha,\beta}[F] = Q_{\alpha,\beta}[F]$. If, moreover, $\alpha(\lambda_{k+1}) \sim \alpha(\lambda_k)$ and $\kappa_k[F] \nearrow +\infty$ as $k_0 \leq k \rightarrow \infty$ then $\lambda_{\alpha,\beta}[F] = q_{\alpha,\beta}[F]$.*

If we choose $\alpha(x) = \ln \ln \gamma(x)$ and $\beta(x) = \ln^+ x$ then $\eta^* = Q_{\alpha,\beta}[F]$, $\eta_* = q_{\alpha,\beta}[F]$ and equalities (2) and (3) can be written, respectively, in the form

$$\overline{\lim}_{\sigma \rightarrow +\infty} (\exp\{\alpha(\ln M(q\sigma, F))\} - p \exp\{\alpha(\ln M(\sigma, F))\}) = +\infty \quad (4)$$

and

$$\underline{\lim}_{\sigma \rightarrow +\infty} (\exp\{\alpha(\ln M(q\sigma, F))\} - p \exp\{\alpha(\ln M(\sigma, F))\}) = -\infty. \quad (5)$$

We remark also that Theorem C implies the following statement.

Lemma 2. *Let $q > 1$, $p > 1$, $\alpha \in L$ and Φ be a positive function continuous on $(x_0, +\infty)$ and increasing to $+\infty$. If*

$$\overline{\lim}_{\sigma \rightarrow +\infty} \frac{\alpha(\Phi(\sigma))}{\ln \sigma} > \frac{\ln p}{\ln q}$$

then

$$\overline{\lim}_{\sigma \rightarrow +\infty} (\exp\{\alpha(\Phi(q\sigma))\} - p \exp\{\alpha(\Phi(\sigma))\}) = +\infty,$$

and if

$$\underline{\lim}_{\sigma \rightarrow +\infty} \frac{\alpha(\Phi(\sigma))}{\ln \sigma} < \frac{\ln p}{\ln q}$$

then

$$\underline{\lim}_{\sigma \rightarrow +\infty} (\exp\{\alpha(\Phi(q\sigma))\} - p \exp\{\alpha(\Phi(\sigma))\}) = -\infty.$$

Using Lemmas 1 and 2 we prove the following theorem.

Theorem 1. Let $\sigma_a = +\infty$, $\alpha \in L$ and $\beta \in L^0$. If $Q_{\alpha,\beta}[F] > 0$ then

$$\overline{\lim}_{\sigma \rightarrow +\infty} (\exp\{\alpha(\ln M(\beta^{-1}(\beta(\sigma) + \ln q), F))\} - p \exp\{\alpha(\ln M(\sigma, F))\}) = +\infty. \quad (6)$$

for each $p > 1$ and $q > 1$ such that $Q_{\alpha,\beta}[F] > \ln p / \ln q$.

If $\alpha \in L_{si}$, $\beta \in L^0$ and $\frac{d\beta^{-1}(c\alpha(x))}{d \ln x} = O(1)$ as $x \rightarrow +\infty$, $\ln k = o(\lambda_k \beta^{-1}(c\alpha(\lambda_k)))$ as $k \rightarrow \infty$ for each $c \in (0, +\infty)$, $\alpha(\lambda_{k+1}) \sim \alpha(\lambda_k)$, $\kappa_k[F] \nearrow +\infty$ as $k_0 \leq k \rightarrow \infty$ and $q_{\alpha,\beta}[F] < +\infty$ then

$$\underline{\lim}_{\sigma \rightarrow +\infty} (\exp\{\alpha(\ln M(\beta^{-1}(\beta(\sigma) + \ln q), F))\} - p \exp\{\alpha(\ln M(\sigma, F))\}) = -\infty \quad (7)$$

for each $p > 1$ and $q > 1$ such that $q_{\alpha,\beta}[F] < \ln p / \ln q$.

Proof. If we put $\Phi(\sigma) = \Phi_1(\beta^{-1}(\ln \sigma))$ then

$$\begin{aligned} & \overline{\lim}_{\sigma \rightarrow +\infty} (\exp\{\alpha(\Phi_1(\beta^{-1}(\beta(\sigma) + \ln q)))\} - p \exp\{\alpha(\Phi_1(\sigma))\}) = \\ &= \overline{\lim}_{\sigma \rightarrow +\infty} (\exp\{\alpha(\Phi_1(\beta^{-1}(\ln(qe^{\beta(\sigma)}))))\} - p \exp\{\alpha(\Phi_1(\beta^{-1}(\ln(e^{\beta(\sigma)}))))\}) = \\ &= \overline{\lim}_{x \rightarrow +\infty} (\exp\{\alpha(\Phi_1(\beta^{-1}(\ln(qx))))\} - p \exp\{\alpha(\Phi_1(\beta^{-1}(\ln x)))\}) = \\ &= \overline{\lim}_{\sigma \rightarrow +\infty} (\exp\{\alpha(\Phi(q\sigma))\} - p \exp\{\alpha(\Phi(\sigma))\}), \end{aligned}$$

$$\begin{aligned} & \underline{\lim}_{\sigma \rightarrow +\infty} (\exp\{\alpha(\Phi_1(\beta^{-1}(\beta(\sigma) + \ln q)))\} - p \exp\{\alpha(\Phi_1(\sigma))\}) = \\ &= \underline{\lim}_{\sigma \rightarrow +\infty} (\exp\{\alpha(\Phi(q\sigma))\} - p \exp\{\alpha(\Phi(\sigma))\}), \end{aligned}$$

$$\overline{\lim}_{\sigma \rightarrow +\infty} \frac{\alpha(\Phi(\sigma))}{\ln \sigma} = \overline{\lim}_{\sigma \rightarrow +\infty} \frac{\alpha(\Phi_1(\beta^{-1}(\ln \sigma)))}{\ln \sigma} = \overline{\lim}_{x \rightarrow +\infty} \frac{\alpha(\Phi_1(\beta^{-1}(x)))}{x} = \overline{\lim}_{\sigma \rightarrow +\infty} \frac{\alpha(\Phi_1(\sigma))}{\beta(\sigma)}$$

and

$$\underline{\lim}_{\sigma \rightarrow +\infty} \frac{\alpha(\Phi(\sigma))}{\ln \sigma} = \underline{\lim}_{\sigma \rightarrow +\infty} \frac{\alpha(\Phi_1(\sigma))}{\beta(\sigma)}.$$

Therefore, in view of Lemma 2 if Φ_1 is a positive function continuous on $(x_0, +\infty)$ and increasing to $+\infty$ and

$$\overline{\lim}_{\sigma \rightarrow +\infty} \frac{\alpha(\Phi_1(\sigma))}{\beta(\sigma)} > \frac{\ln p}{\ln q}$$

then

$$\overline{\lim}_{\sigma \rightarrow +\infty} (\exp\{\alpha(\Phi_1(\beta^{-1}(\beta(\sigma) + \ln q)))\} - p \exp\{\alpha(\Phi_1(\sigma))\}) = +\infty,$$

and if

$$\underline{\lim}_{\sigma \rightarrow +\infty} \frac{\alpha(\Phi_1(\sigma))}{\beta(\sigma)} < \frac{\ln p}{\ln q}$$

then

$$\underline{\lim}_{\sigma \rightarrow +\infty} (\exp\{\alpha(\Phi_1(\beta^{-1}(\beta(\sigma) + \ln q)))\} - p \exp\{\alpha(\Phi_1(\sigma))\}) = -\infty$$

Finally, we choose $\Phi_1(\sigma) = \ln M(\sigma, F)$. Then from hence it follows that if

$$\overline{\lim}_{\sigma \rightarrow +\infty} \frac{\alpha(\ln M(\sigma, F))}{\beta(\sigma)} > \frac{\ln p}{\ln q}, \quad (8)$$

then (6) holds and if

$$\underline{\lim}_{\sigma \rightarrow +\infty} \frac{\alpha(\ln M(\sigma, F))}{\beta(\sigma)} < \frac{\ln p}{\ln q} \quad (9)$$

then (7) holds.

If $Q_{\alpha, \beta}[F] > 0$ then for every $Q \in (0, Q_{\alpha, \beta}[F])$ there exists an increasing to ∞ sequence (k_n) such that $\ln |f_{k_n}| \geq -\lambda_{k_n} \beta^{-1}(\alpha(\lambda_{k_n})/Q)$ for all n . We choose $\sigma_n = \beta^{-1}(\alpha(\lambda_{k_n})/Q) + 1$. Then by the Cauchy inequality

$$\begin{aligned} \ln M(\sigma_n, F) &\geq \ln |f_{k_n}| + \sigma_n \lambda_{k_n} \geq -\lambda_{k_n} \beta^{-1}(\alpha(\lambda_{k_n})/Q) + \lambda_{k_n} (\beta^{-1}(\alpha(\lambda_{k_n})/Q) + 1) = \\ &= \lambda_{k_n} = \alpha^{-1}(Q\beta(\sigma_n - 1)), \end{aligned}$$

whence in view of the condition $\beta \in L^0$ and of the arbitrariness of Q we get

$$\overline{\lim}_{\sigma \rightarrow +\infty} \frac{\alpha(\ln M(\sigma, F))}{\beta(\sigma)} \geq \overline{\lim}_{n \rightarrow \infty} \frac{\alpha(\ln M(\sigma_n, F))}{\beta(\sigma_n)} \geq Q_{\alpha, \beta}[F].$$

Therefore, if $Q_{\alpha, \beta}[F] > \ln p / \ln q$ then (8) and, thus, (6) hold. The first part of Theorem 1 is proved.

If $q_{\alpha, \beta}[F] < +\infty$ then by Lemma 1

$$\underline{\lim}_{\sigma \rightarrow +\infty} \frac{\alpha(\ln M(\sigma, F))}{\beta(\sigma)} = \lambda_{\alpha, \beta}[F] = q_{\alpha, \beta}[F].$$

Therefore, if $q_{\alpha, \beta}[F] < \ln p / \ln q$ then (9) and, thus, (7) hold. The proof of Theorem 1 is complete. $\square$

We remark that if $\beta(x) = \ln^+ x$ then (6) and (7) imply (4) and (5).

If we choose $\alpha(x) = \ln^+ x$ and $\beta(x) = x^+$ then we get the following result in the R-order scale.

Corollary 1. *Let $\sigma_a = +\infty$. If $Q_R[F] := \overline{\lim}_{k \rightarrow \infty} \frac{\lambda_k \ln \lambda_k}{-\ln |f_k|} > 0$ then*

$$\overline{\lim}_{\sigma \rightarrow +\infty} (\ln M(\sigma + \ln q, F) - p \ln M(\sigma, F)) = +\infty$$

for each $p > 1$ and $q > 1$ such that $Q_R[F] > \ln p / \ln q$.

If $\ln k = o(\lambda_k \ln \lambda_k)$, $\ln \lambda_{k+1} \sim \ln \lambda_k$, $\kappa_k[F] \nearrow +\infty$ as $k_0 \leq k \rightarrow \infty$ and $q_R[F] := \underline{\lim}_{k \rightarrow \infty} \frac{\lambda_k \ln \lambda_k}{-\ln |f_k|} < +\infty$ then

$$\underline{\lim}_{\sigma \rightarrow +\infty} (\ln M(\sigma + \ln q, F) - p \ln M(\sigma, F)) = -\infty$$

for each $p > 1$ and $q > 1$ such that $q_R[F] < \ln p / \ln q$.

2 DIRICHLET SERIES ABSOLUTELY CONVERGENT IN A HALF-PLANE

If $\sigma_a[F] = 0$, $\alpha \in L$ and $\beta \in L$ then the quantities

$$\varrho_{\alpha,\beta}^{(0)}[F] := \overline{\lim}_{\sigma \uparrow 0} \frac{\alpha(\ln M(\sigma, F))}{\beta(1/|\sigma|)}, \quad \lambda_{\alpha,\beta}^{(0)}[F] := \lim_{\sigma \uparrow 0} \frac{\alpha(\ln M(\sigma, F))}{\beta(1/|\sigma|)}$$

are called [3], [4] the generalized (α, β) -order and the generalized lower (α, β) -order of F accordingly. We put

$$Q_{\alpha,\beta}^{(0)}[F] = \overline{\lim}_{k \rightarrow \infty} \frac{\alpha(\lambda_k)}{\beta(\lambda_k / \ln^+ |f_k|)}, \quad q_{\alpha,\beta}^{(0)}[F] = \lim_{k \rightarrow \infty} \frac{\alpha(\lambda_k)}{\beta(\lambda_k / \ln^+ |f_k|)}.$$

The following lemma is correct [3], [4].

Lemma 3. Let $\alpha \in L_{si}$, $\beta \in L_{si}$ and

$$\frac{x}{\beta^{-1}(c\alpha(x))} \uparrow +\infty, \quad \alpha\left(\frac{x}{\beta^{-1}(c\alpha(x))}\right) = (1 + o(1))\alpha(x) \quad (10)$$

as $x_0(c) \leq x \rightarrow +\infty$ for each $c \in (0, +\infty)$. Suppose that $\sigma_a[F] = 0$ and $\ln k = o(\lambda_k / \beta^{-1}(c\alpha(\lambda_k)))$ as $k \rightarrow \infty$ for each $c \in (0, +\infty)$. Then $\varrho_{\alpha,\beta}^{(0)}[F] = Q_{\alpha,\beta}^{(0)}[F]$. If, moreover, $\alpha(\lambda_{k+1}) \sim \alpha(\lambda_k)$ and $\kappa_k[F] \nearrow 0$ as $k_0 \leq k \rightarrow \infty$ then $\lambda_{\alpha,\beta}^{(0)}[F] = q_{\alpha,\beta}^{(0)}[F]$.

Using Lemmas 2 and 3 we prove the following theorem.

Theorem 2. Let $\sigma_a = 0$, $\beta \in L_{si}$, $\alpha(e^x) \in L_{si}$ and $\alpha(x) = o(\beta(x))$ as $x \rightarrow +\infty$. If $Q_{\alpha,\beta}^{(0)}[F] > 0$ then

$$\overline{\lim}_{\sigma \uparrow 0} \left(\exp \left\{ \alpha \left(\ln M \left(-\frac{1}{\beta^{-1}(\beta(1/|\sigma|) + \ln q)}, F \right) \right) \right\} - p \exp \{ \alpha(\ln M(\sigma, F)) \} \right) = +\infty. \quad (11)$$

for each $p > 1$ and $q > 1$ such that $Q_{\alpha,\beta}^{(0)}[F] > \ln p / \ln q$.

If the functions $\alpha \in L_{si}$ and $\beta \in L_{si}$ satisfy conditions (10), $\ln k = o(\lambda_k / \beta^{-1}(c\alpha(\lambda_k)))$ as $k \rightarrow \infty$ for each $c \in (0, +\infty)$, $\alpha(\lambda_{k+1}) \sim \alpha(\lambda_k)$, $\kappa_k[F] \nearrow 0$ as $k_0 \leq k \rightarrow \infty$ and $q_{\alpha,\beta}^{(0)}[F] < +\infty$ then

$$\lim_{\sigma \uparrow 0} \left(\exp \left\{ \alpha \left(\ln M \left(-\frac{1}{\beta^{-1}(\beta(1/|\sigma|) + \ln q)}, F \right) \right) \right\} - p \exp \{ \alpha(\ln M(\sigma, F)) \} \right) = -\infty \quad (12)$$

for each $p > 1$ and $q > 1$ such that $q_{\alpha,\beta}^{(0)}[F] < \ln p / \ln q$.

Proof. If we put $\Phi(x) = \Phi_1(-1/\beta^{-1}(\ln x))$ then

$$\begin{aligned} & \overline{\lim}_{\sigma \uparrow 0} \left(\exp \left\{ \alpha \left(\Phi_1 \left(-\frac{1}{\beta^{-1}(\beta(1/|\sigma|) + \ln q)} \right) \right) \right\} - p \exp \{ \alpha(\Phi_1(\sigma)) \} \right) = \\ & = \overline{\lim}_{\sigma \uparrow 0} \left(\exp \left\{ \alpha \left(\Phi_1 \left(\frac{-1}{\beta^{-1}(\ln(qe^{\beta(1/|\sigma|)}))} \right) \right) \right\} - p \exp \left\{ \alpha \left(\Phi_1 \left(\frac{-1}{\beta^{-1}(\ln e^{\beta(1/|\sigma|)})} \right) \right) \right\} \right) = \\ & = \overline{\lim}_{x \rightarrow +\infty} \left(\exp \left\{ \alpha \left(\Phi_1 \left(-\frac{1}{\beta^{-1}(\ln(qx))} \right) \right) \right\} - p \exp \left\{ \alpha \left(\Phi_1 \left(-\frac{1}{\beta^{-1}(\ln x)} \right) \right) \right\} \right) = \\ & = \overline{\lim}_{x \rightarrow +\infty} (\exp \{ \alpha(\Phi(qx)) \} - p \exp \{ \alpha(\Phi(x)) \}), \end{aligned}$$

$$\begin{aligned} \lim_{\sigma \uparrow 0} \left(\exp \left\{ \alpha \left(\Phi_1 \left(-\frac{1}{\beta^{-1}(\beta(1/|\sigma|) + \ln q)} \right) \right) \right\} - p \exp \{ \alpha(\Phi_1(\sigma)) \} \right) = \\ = \lim_{x \rightarrow +\infty} (\exp \{ \alpha(\Phi(qx)) \} - p \exp \{ \alpha(\Phi(x)) \}), \end{aligned}$$

$$\lim_{x \rightarrow +\infty} \frac{\alpha(\Phi(x))}{\ln x} = \lim_{x \rightarrow +\infty} \frac{\alpha(\Phi_1(-1/\beta^{-1}(\ln x)))}{\ln x} = \lim_{\sigma \uparrow 0} \frac{\alpha(\Phi_1(\sigma))}{\beta(1/|\sigma|)}$$

and

$$\lim_{x \rightarrow +\infty} \frac{\alpha(\Phi(x))}{\ln x} = \lim_{\sigma \uparrow 0} \frac{\alpha(\Phi_1(\sigma))}{\beta(1/|\sigma|)}$$

Therefore, in view of Lemma 2 if Φ_1 is a positive function continuous on $(x_0, 0)$ and increasing to $+\infty$ and

$$\lim_{\sigma \uparrow 0} \frac{\alpha(\Phi_1(\sigma))}{\beta(1/|\sigma|)} > \frac{\ln p}{\ln q}$$

then

$$\lim_{\sigma \uparrow 0} \left(\exp \left\{ \alpha \left(\Phi_1 \left(-\frac{1}{\beta^{-1}(\beta(1/|\sigma|) + \ln q)} \right) \right) \right\} - p \exp \{ \alpha(\Phi_1(\sigma)) \} \right) = +\infty$$

and if

$$\lim_{\sigma \uparrow 0} \frac{\alpha(\Phi_1(\sigma))}{\beta(1/|\sigma|)} < \frac{\ln p}{\ln q}$$

then

$$\lim_{\sigma \uparrow 0} \left(\exp \left\{ \alpha \left(\Phi_1 \left(-\frac{1}{\beta^{-1}(\beta(1/|\sigma|) + \ln q)} \right) \right) \right\} - p \exp \{ \alpha(\Phi_1(\sigma)) \} \right) = -\infty$$

Finally, we choose $\Phi_1(\sigma) = \ln M(\sigma, F)$. Then from hence it follows that if

$$\lim_{\sigma \uparrow 0} \frac{\alpha(\ln M(\sigma, F))}{\beta(1/|\sigma|)} > \frac{\ln p}{\ln q} \quad (13)$$

then (11) holds, and if

$$\lim_{\sigma \uparrow 0} \frac{\alpha(\ln M(\sigma, F))}{\beta(1/|\sigma|)} < \frac{\ln p}{\ln q} \quad (14)$$

then (12) holds.

If $Q_{\alpha, \beta}^{(0)}[F] > 0$ then for every $Q \in (0, Q_{\alpha, \beta}[F])$ there exists an increasing to ∞ sequence (k_n) such that $\ln^+ |f_{k_n}| \geq \frac{\lambda_{k_n}}{\beta^{-1}(\alpha(\lambda_{k_n})/Q)}$ for all n . We choose $\sigma_n = -\frac{1}{2\beta^{-1}(\alpha(\lambda_{k_n})/Q)}$. Then by the Cauchy inequality

$$\begin{aligned} \ln M(\sigma_n, F) &\geq \ln |f_{k_n}| + \sigma_n \lambda_{k_n} \geq \lambda_{k_n} \left(\frac{1}{\beta^{-1}(\alpha(\lambda_{k_n})/Q)} - \frac{1}{2\beta^{-1}(\alpha(\lambda_{k_n})/Q)} \right) = \\ &= \frac{\lambda_{k_n}}{2\beta^{-1}(\alpha(\lambda_{k_n})/Q)} = \lambda_{k_n} |\sigma_n| = |\sigma_n| \alpha^{-1} \left(Q \beta \left(\frac{1}{2|\sigma_n|} \right) \right), \end{aligned}$$

whence in view of the conditions $\beta \in L_{si}$, $\alpha(e^x) \in L_{si}$ and $\alpha(x) = o(\beta(x))$ as $x \rightarrow +\infty$ we get

$$\begin{aligned} Q(1+o(1))\beta(1/|\sigma_n|) &\leq \alpha(\ln M(\sigma_n, F)/|\sigma_n|) = \alpha(\exp\{\ln \ln M(\sigma_n, F) + \ln(1/|\sigma_n|)\}) \leq \\ &\leq \alpha(\exp\{2 \max\{\ln \ln M(\sigma_n, F), \ln(1/|\sigma_n|)\}\}) = \\ &= (1+o(1))\alpha(\exp\{\max\{\ln \ln M(\sigma_n, F), \ln(1/|\sigma_n|)\}\}) = \\ &= (1+o(1)) \max\{\alpha(\ln M(\sigma_n, F)), \alpha(1/|\sigma_n|)\} \leq \\ &\leq (1+o(1))(\alpha(\ln M(\sigma_n, F)) + \alpha(1/|\sigma_n|)) = \\ &= (1+o(1))\alpha(\ln M(\sigma_n, F)) + o(\beta(1/|\sigma_n|)), \quad \sigma \uparrow 0. \end{aligned}$$

Therefore, in view of the arbitrariness of Q we have

$$\overline{\lim}_{\sigma \uparrow 0} \frac{\alpha(\ln M(\sigma, F))}{\beta(1/|\sigma|)} \geq \overline{\lim}_{n \rightarrow \infty} \frac{\alpha(\ln M(\sigma_n, F))}{\beta(1/|\sigma_n|)} \geq Q_{\alpha, \beta}^{(0)}[F], \quad (15)$$

and if $Q_{\alpha, \beta}^{(0)}[F] > \ln p / \ln q$ then (13) and, thus, (11) hold. The first part of Theorem 2 is proved.

If $q_{\alpha, \beta}^{(0)}[F] < +\infty$ then by Lemma 3

$$\lim_{\sigma \uparrow 0} \frac{\alpha(\ln M(\sigma, F))}{\beta(1/|\sigma|)} = \lambda_{\alpha, \beta}^{(0)}[F] = q_{\alpha, \beta}^{(0)}[F].$$

Therefore, if $q_{\alpha, \beta}^{(0)}[F] < \ln p / \ln q$ then (14) and, thus, (12) hold. The proof of Theorem 2 is complete. $\square$

From conditions of Theorem 2 it follows that the function α grows more slowly than the function β . In the case if the function β grows more slowly than the function α , the following theorem is true.

Theorem 3. Let $\sigma_a = 0$, $\beta \in L$ and $\alpha \in L^0$. If $P_{\alpha, \beta}^{(0)}[F] := \overline{\lim}_{n \rightarrow \infty} \frac{\alpha(\ln |f_n|)}{\beta(\lambda_n)} > 0$ then (11)

holds for each $p > 1$ and $q > 1$ such that $P_{\alpha, \beta}^{(0)}[F] > \ln p / \ln q$.

If the functions $\alpha \in L_{si}$ and $\beta \in L_{si}$ satisfy conditions

$$\frac{x}{\alpha^{-1}(c\beta(x))} \uparrow +\infty, \quad \alpha\left(\frac{x}{\gamma^{-1}(c\beta(x))}\right) = (1+o(1))\beta(x), \quad x \rightarrow +\infty. \quad (16)$$

for each $c \in (0, +\infty)$, $\gamma(\ln n) = o(\beta(\lambda_n))$, $\beta(\lambda_{n+1}) \sim \beta(\lambda_n)$ and $\kappa_n[F] \nearrow 0$ as $n_0 \leq n \rightarrow \infty$ and $p_{\alpha, \beta}^{(0)}[F] := \lim_{n \rightarrow \infty} \frac{\alpha(\ln |f_n|)}{\beta(\lambda_n)} < +\infty$ then (12) holds for each $p > 1$ and $q > 1$ such that $p_{\alpha, \beta}^{(0)}[F] < \ln p / \ln q$.

Proof. If $P_{\alpha, \beta}^{(0)}[F] > 0$ then for every $P \in (0, P_{\alpha, \beta}^{(0)}[F])$ there exists an increasing to ∞ sequence (k_n) such that $\ln |f_{k_n}| \geq \alpha^{-1}(P\beta(\lambda_{k_n}))$ for all n . We choose $\sigma_n = -1/\lambda_{k_n}$. Then by the Cauchy inequality

$$\ln M(\sigma_n, F) \geq \alpha^{-1}(P\beta(\lambda_{k_n})) + \sigma_n \lambda_{k_n} = \alpha^{-1}(P\beta(\lambda_{k_n})) - 1 = \alpha^{-1}(P\beta(1/|\sigma_n|)) - 1,$$

whence we obtain (15) with $P_{\alpha,\beta}^{(0)}[F]$ instead $Q_{\alpha,\beta}^{(0)}[F]$, and thus, the first part of Theorem 3 is proved.

If the functions $\alpha \in L_{si}$ and $\beta \in L_{si}$ satisfy conditions (16), $\gamma(\ln n) = o(\beta(\lambda_n))$, $\beta(\lambda_{n+1}) \sim \beta(\lambda_n)$ and $\kappa_n[F] \nearrow 0$ as $n_0 \leq n \rightarrow \infty$ then [4] $\lim_{\sigma \uparrow 0} \frac{\alpha(\ln M(\sigma, F))}{\beta(1/|\sigma|)} = p_{\alpha,\beta}^{(0)}[F]$ and as above we get the correctness of the second part of Theorem 3. $\square$

As a conclusion, we present two statements corresponding to scales of finite order and finite R-order.

Proposition 1. *Let $\sigma_a = 0$. If $\overline{\lim}_{n \rightarrow \infty} \frac{\ln^+ \ln |f_n|}{\ln \lambda_n} = \tau \in (0, 1)$ then*

$$\overline{\lim}_{\sigma \uparrow 0} (\ln M(\sigma/q, F) - p \ln M(\sigma, F)) = +\infty.$$

for each $p > 1$ and $q > 1$ such that $\tau/(1 - \tau) > \ln p / \ln q$.

If $\ln \ln n = o(\ln \lambda_n)$, $\ln \lambda_{n+1} \sim \ln \lambda_n$, $\kappa_n[F] \nearrow 0$ as $n_0 \leq n \rightarrow \infty$ and $\overline{\lim}_{n \rightarrow \infty} \frac{\ln^+ \ln |f_n|}{\ln \lambda_n} = \eta < 1$ then

$$\underline{\lim}_{\sigma \uparrow 0} (\ln M(\sigma/q, F) - p \ln M(\sigma, F)) = -\infty.$$

for each $p > 1$ and $q > 1$ such that $\eta/(1 - \eta) < \ln p / \ln q$.

Proof. If $\tau > 0$ then for every $\tau^0 \in (0, \tau)$ there exists an increasing to ∞ sequence (k_n) such that $\ln |f_{k_n}| \geq \lambda_{k_n}^{\tau^0}$ for all n . We choose $\sigma_n = -(1/2)\lambda_{k_n}^{\tau^0-1}$. Then by the Cauchy inequality

$$\ln M(\sigma_n, F) \geq \lambda_{k_n}^{\tau^0} - \frac{1}{2} \lambda_{k_n}^{\tau^0} = \frac{1}{2} \lambda_{k_n}^{\tau^0} = \frac{1}{2} \left(\frac{1}{2|\sigma|} \right)^{\tau^0/(1-\tau^0)},$$

i. e. in view the arbitrariness of τ^0 we have

$$\overline{\lim}_{\sigma \uparrow 0} \frac{\ln \ln M(\sigma, F)}{\ln(1/|\sigma|)} \geq \frac{\tau^0}{1 - \tau^0} > 0$$

whence we obtain (15) with $\alpha(x) = \beta(x) = \ln^+ x$ and $\tau^0/(1 - \tau^0)$ instead $Q_{\alpha,\beta}^{(0)}[F]$, and thus, the first part of Proposition 1 is proved.

To prove the second part, it is enough to note that if $\ln \ln n = o(\ln \lambda_n)$, $\ln \lambda_{n+1} \sim \ln \lambda_n$, $\kappa_n[F] \nearrow 0$ as $n_0 \leq n \rightarrow \infty$ and $\eta < 1$ then [1] $\lim_{\sigma \uparrow 0} \frac{\ln \ln M(\sigma, F)}{\ln(1/|\sigma|)} \geq \frac{\eta}{1 - \eta} < +\infty$. $\square$

Proposition 2. *Let $\sigma_a = 0$. If $\overline{\lim}_{k \rightarrow \infty} \frac{\ln \lambda_k}{\lambda_k} \ln^+ |f_k| = Q_R > 0$ then*

$$\overline{\lim}_{\sigma \uparrow 0} (\ln M(\sigma + \ln q, F) - p \ln M(\sigma, F)) = +\infty.$$

for each $p > 1$ and $q > 1$ such that $Q_R > \ln p / \ln q$.

If $\overline{\lim}_{k \rightarrow \infty} \frac{\ln \ln k}{\ln \lambda_k} < 1$, $\ln \lambda_{k+1} \sim \ln \lambda_k$, $\kappa_k[F] \nearrow 0$ as $k_0 \leq k \rightarrow \infty$ and $\varliminf_{k \rightarrow \infty} \frac{\ln \lambda_k}{\lambda_k} \ln^+ |f_k| = q_R < +\infty$ then

$$\varliminf_{\sigma \uparrow 0} (\ln M(\sigma + \ln q, F) - p \ln M(\sigma, F)) = -\infty.$$

for each $p > 1$ and $q > 1$ such that $q_R < \ln p / \ln q$.

Proof. If $Q_R > 0$ then for every $Q \in (0, Q_R)$ there exists an increasing to ∞ sequence (k_n) such that $\ln |f_{k_n}| \geq Q \lambda_{k_n} / \ln \lambda_{k_n}$ for all n . We choose $\sigma_n = -\xi / \ln \lambda_{k_n}$, where $0 < \xi < Q$. Then by the Cauchy inequality

$$\ln M(\sigma_n, F) \geq \lambda_{k_n} \left(\frac{Q}{\ln \lambda_{k_n}} + \sigma \right) = \frac{(Q - \xi) \lambda_{k_n}}{\ln \lambda_{k_n}} = \frac{Q - \xi}{\xi} |\sigma| \exp \left\{ \frac{\xi}{|\sigma|} \right\}$$

i. e. in view the arbitrariness of Q and ξ we have

$$\overline{\lim}_{\sigma \uparrow 0} |\sigma| \ln \ln M(\sigma, F) \geq Q_R > 0$$

whence we obtain (15) with $\alpha(x) = \ln^+ x$, $\beta(x) = x^+$ and Q_R instead $Q_{\alpha, \beta}^{(0)}[F]$, and thus, the first part of Proposition 2 is proved.

To prove the second part, it is enough to note that if $\overline{\lim}_{k \rightarrow +\infty} \frac{\ln \ln k}{\ln \lambda_k} < 1$, $\ln \lambda_{k+1} \sim \ln \lambda_k$ and $\kappa_k[F] \nearrow 0$ as $k_0 \leq k \rightarrow \infty$ then [2] $\varliminf_{\sigma \uparrow 0} |\sigma| \ln \ln M(\sigma, F) = q_R < +\infty$. $\square$

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Шеремета М.М., Трухан Ю.С. *Про зростання максимуму модуля рядів Діріхле* // Буковинський матем. журнал — 2024. — Т.12, №1. — С. 32–42.

Для ряду Діріхле $F(s) = \sum_{n=0}^{\infty} f_n \exp\{s\lambda_n\}$ з невід'ємними зростаючими $+\infty$ показниками λ_n і абсцисою абсолютної збіжності $\sigma_a \in (-\infty, +\infty]$ вивчено зв'язок між зростанням на $(-\infty, \sigma_a)$ максимуму модуля $M(\sigma, F) = \sup\{|F(\sigma + it)| : t \in \mathbb{R}\}$ і поведінкою коефіцієнтів f_n . Для цього через L позначено клас неперервних зростаючих до $+\infty$ на $(x_0, +\infty)$ функцій α . Належність α до класу L^0 означає, що $\alpha \in L$ і $\alpha((1+o(1))x) = (1+o(1))\alpha(x)$ при $x \rightarrow +\infty$, а $\alpha \in L_{si}$, якщо $\alpha \in L$ і $\alpha(cx) = (1+o(1))\alpha(x)$ при $x \rightarrow +\infty$.

Для цілих рядів Діріхле ($\sigma_a = +\infty$), наприклад, доведено, що якщо $\alpha \in L$, $\beta \in L^0$, то $\overline{\lim}_{\sigma \rightarrow +\infty} (\exp\{\alpha(\ln M(\beta^{-1}(\beta(\sigma) + \ln q), F))\} - p \exp\{\alpha(\ln M(\sigma, F))\}) = +\infty$ для таких $p > 1$ і $q > 1$, що $\overline{\lim}_{n \rightarrow \infty} \alpha(\lambda_n)/\beta(\lambda_n^{-1} \ln(1/|f_n|)) > \ln p / \ln q$. Якщо ж $\alpha \in L_{si}$, $\beta \in L^0$, $\frac{d\beta^{-1}(c\alpha(x))}{d \ln x} = O(1)$ при $x \rightarrow +\infty$ і $\ln n = o(\lambda_n \beta^{-1}(c\alpha(\lambda_n)))$ при $n \rightarrow \infty$ для кожного $c \in (0, +\infty)$, $\alpha(\lambda_{n+1}) \sim \alpha(\lambda_n)$ при $n \rightarrow \infty$ і $\frac{\ln |f_n| - \ln |f_{n+1}|}{\lambda_{n+1} - \lambda_n} \nearrow +\infty$ при $n_0 \leq n \rightarrow \infty$, то $\overline{\lim}_{\sigma \rightarrow +\infty} (\exp\{\alpha(\ln M(\beta^{-1}(\beta(\sigma) + \ln q), F))\} - p \exp\{\alpha(\ln M(\sigma, F))\}) = -\infty$ для таких $p > 1$ і $q > 1$, що $\underline{\lim}_{n \rightarrow \infty} \alpha(\lambda_n)/\beta(\lambda_n^{-1} \ln(1/|f_n|)) < \ln p / \ln q$.

Подібні результати отримано для рядів Діріхле, абсолютно збіжних у півплощині $\{s : \text{Res} < 0\}$. Наприклад, доведено, що якщо $\sigma_a = 0$, $\beta \in L_{si}$, $\alpha(e^x) \in L_{si}$ і $\alpha(x) = o(\beta(x))$ при $x \rightarrow +\infty$, то

$$\overline{\lim}_{\sigma \uparrow 0} \left(\exp \left\{ \alpha \left(\ln M \left(-\frac{1}{\beta^{-1}(\beta(1/|\sigma|) + \ln q)}, F \right) \right) \right\} - p \exp \{ \alpha(\ln M(\sigma, F)) \} \right) = +\infty$$

для таких $p > 1$ і $q > 1$, що $\overline{\lim}_{n \rightarrow \infty} \frac{\alpha(\lambda_n)}{\beta(\lambda_n / \ln^+ |f_n|)} > \ln p / \ln q$.