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INTEGRAL REPRESENTATION OF HYPERBOLICALLY CONVEX FUNCTIONS

An article consists of two parts.

In the first part the sufficient and necessary conditions for an integral representation of hyperbolically convex (h.c.) functions $k(x)$ $\big(x \in \mathbb{R}^\infty = \mathbb{R}^1 \times \mathbb{R}^1 \times \dots\big)$ are proved. For this purpose in $\mathbb{R}^\infty$ we introduce measures $\omega_1(x)$, $\omega_{\frac{1}{2}}(x)$. The positive definiteness of a function will be understood on the integral sense with respect to the measure $\omega_1(x)$. Then we proved that the measure $\rho(\lambda)$ in the integral representation is concentrated on $l_2^+ = \Big\{\lambda \in \mathbb{R}_+^\infty = \mathbb{R}_+^1 \times \mathbb{R}_+^1 \times \dots \Big| \sum\limits_{n=1}^{\infty} \lambda_n^2 < \infty\Big\}$. The equality for $k(x)$ $(x \in \mathbb{R}^\infty)$ is regarded as an equality for almost all $x \in \mathbb{R}^\infty$ with respect to measure $\omega_{\frac{1}{2}}(x)$.

In the second part we proved the sufficient and necessary conditions for integral representation of h.c. functions $k(x)$ $\big(x \in \mathbb{R}_0^\infty$ is a nuclear space$\big)$. The positive definiteness of a function $k(x)$ will be understood on the pointwise sense. For this purpose we shall construct a rigging (chain) $\mathbb{R}_0^\infty \subset l_2 \subset \mathbb{R}^\infty$. Then, given that the projection and inductive topologies are coinciding, we shall obtaine the integral representation for $k(x)$ $(x \in \mathbb{R}_0^\infty)$

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INTEGRAL REPRESENTATION OF HYPERBOLICALLY CONVEX FUNCTIONS

1 Introduction

During a last decade an infinite-dimensional analysis has been developing rapidly. With the help of methods of a spectral theory of operators Yu. M. Berezansky obtained the integral representation for the positively definite functions [2]. These methods of obtaining of the integral representation for another positively definite kernels had been used at [3, 5, 7, 8, 9, 11].

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This article presents the integral representation for a class of evenly positive definite (e.p.d) functions, and namely hyperbolically convex (h.c) functions.

The study of these integral representation for h.c. functions is useful from different point of view: on the one hand we can prove the theorem of type Stoune's. For this, the method [4] can be used and obtain the integral representation for a family (A_t) $(t \in \mathbb{R}^1)$ of self-adjoint unbounded operators, in the nuclear space, satisfy following conditions:

1) $\frac{1}{2}\left[A_{t+s} + A_{t-s}\right] = A_t A_s;\ A_t = A_{-t};\ A_0 = I.$

2) $A_{\left(\frac{t+s}{2}\right)} = \frac{1}{2}\left[A_t + A_s\right].$

On the other hand, such functions appear in applications, for example, in the description of various models of physical systems with infinitely many degrees of freedom [6].

2 Hyperbolically convex functions of infinite number of variables

Let $\mathbb{R}^\infty = \mathbb{R}^1 \times \mathbb{R}^1 \times \ldots$ be a space with a sequences $x = (x_j)_{j=1}^\infty$, $x_j \in \mathbb{R}^1$. We introduce a Gaussian measures in this space $d\omega_1(x) = (p(x_1)dx_1) \otimes (p(x_2)dx_2) \otimes \ldots$, where $p(t) = \pi^{-\frac{1}{2}} e^{-t^2} dt$, $t \in \mathbb{R}^1$ and $d\omega_{\frac{1}{2}}(x) = (p_0(x_1)dx_1) \otimes (p_0(x_2)dx_2) \otimes \ldots$, where $p_0(t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt$, $t \in \mathbb{R}^1$. Then, if $f(x)$ is measurable and sumable with respect to $d\omega_{\frac{1}{2}}(x)$, then $\frac{1}{2}\left[f(x+y) + f(x-y)\right]$ is measurable and sumable with respect to $d\omega_1(x) \otimes \omega_1(y)$, moreover

$$\int\limits_{\mathbb{R}^\infty} \int\limits_{\mathbb{R}^\infty} \frac{1}{2}\left[f(x+y) + f(x-y)\right] d\omega_1(x)\, d\omega_1(y) = \int\limits_{\mathbb{R}^\infty} f(x)\, d\omega_{\frac{1}{2}}(x). \tag{2.1}$$

A real-valued function $k(x)$ $(x \in \mathbb{R}^\infty)$, which is even for the each variable, is measurable and which satisfies an estimate $k(x) \leq c e^{\sum\limits_{n=1}^\infty N_n x_n^2}$ $\left(c, N_n > 0; \sum\limits_{n=1}^\infty N_n < \infty\right)$ almost everywhere with respect to $d\omega_{\frac{1}{2}}(x)$ is called *h.c.*, if it is convex and for the any cylindrical function $u(x) = u_{\mathrm{C}^\dagger}(x_1, \ldots, x_m)$ $(u_{\mathrm{C}^\dagger} \in C_0^m(\mathbb{R}^m))$ the inequality

$$\int\limits_{\mathbb{R}^\infty} \int\limits_{\mathbb{R}^\infty} \frac{1}{2}\left[k(x+y) + k(x-y)\right] \overline{u(x)} u(y)\, d\omega_1(x)\, d\omega_1(y) \geq 0 \tag{2.2}$$

holds. That is, $k(x)$ is even-positive defined (e.p.d.).

Theorem 1. *In order for the function $k(x)$ $(x \in \mathbb{R}^\infty)$ to admit the integral representation*

$$k(x) = \int\limits_{l_2^+} \prod_{j=1}^\infty \mathrm{Ch}\, \lambda_j x_j\, d\sigma(\lambda), \tag{2.3}$$

where $d\sigma(\lambda)$ is the non-negative finite measure with the Borel σ-algebra of cylindrical sets from l_2^+, it is necessary and sufficient for the function $k(x)$ to be h.c. and e.p.d.. The equality in (2.3) is regarded as equality for almost all $x \in \mathbb{R}^\infty$ with respect to the measure $d\omega_{\frac{1}{2}}(x)$. The measure $d\sigma(\lambda)$ is uniquely determined for the given k. The integral of vector functions $l_2^+ \ni \lambda \to \prod\limits_{j=1}^\infty \mathrm{Ch}\, \lambda_j x_j \in L_2\left(\mathbb{R}^\infty, d\omega_{\frac{1}{2}}(x)\right)$ converges strongly.

Proof. Sufficiency. It is well-known that, for e.p.d. functions on $\mathbb{R}^1$, the following integral representation (see. [1], p. 697)

$$k(x_1)=\int\limits_{\mathbb{R}^1}\operatorname{Cos}\sqrt{\lambda_1}x_1\,d\chi(\lambda_1)=\int\limits_{-\infty}^{0}\operatorname{Cos}\sqrt{\lambda_1}x_1\,d\chi(\lambda_1)+\int\limits_{0}^{\infty}\operatorname{Cos}\sqrt{\lambda_1}x_1\,d\chi(\lambda_1)=$$
$$=\int\limits_{\mathbb{R}^1_+}\operatorname{Ch}\lambda_1x_1\,d\sigma_1(\lambda_1)+\int\limits_{\mathbb{R}^1_+}\operatorname{Cos}\lambda_1x_1\,d\nu_1(\lambda_1),$$

is true. Then if $k(x_1)$ is the convex too, we obtain the following representation

$$k(x_1)=\int\limits_{\mathbb{R}^1_+}\operatorname{Ch}\lambda_1x_1\,d\sigma_1(\lambda_1).$$

Now we prove that for the h.c. and e.p.d. function $k(x_1,\dots,x_n)$, which is given on $\mathbb{R}^n$, the following integral representation is true

$$k(x_1,\dots,x_n)=\int\limits_{\mathbb{R}^n_+}\prod_{j=1}^{n}\operatorname{Ch}\lambda_jx_j\,d\sigma_n(\lambda_1,\dots,\lambda_n). \tag{2.4}$$

In the proof a method of mathematical induction on n is used.

Let $k(x_1,\dots,x_{n-1})=\int\limits_{\mathbb{R}^{n-1}_+}\prod\limits_{j=1}^{n-1}\operatorname{Ch}\lambda_jx_j\,d\sigma_{n-1}(\lambda_1,\dots,\lambda_{n-1})$, and

$$k(x_1,\dots,x_n)=\int\limits_{\mathbb{R}^n}\prod_{j=1}^{n}\operatorname{Cos}\sqrt{\lambda_j}x_j\,d\chi_n(\lambda_1,\dots,\lambda_n).$$

But since $\int\limits_{\mathbb{R}^{n-1}_+}d\chi_n(\lambda_1,\dots,\lambda_n)$ is the measure, concentrated on $\mathbb{R}^{n-1}_+$, then

$$k(x_1,\dots,x_n)=\int\limits_{\mathbb{R}^{n-1}_+}\prod_{j=1}^{n-1}\operatorname{Ch}\lambda_jx_j\int\limits_{\mathbb{R}^1_+}\operatorname{Ch}\lambda_nx_n\,d\sigma_n(\lambda_1,\dots,\lambda_n)+$$
$$+\int\limits_{\mathbb{R}^{n-1}_+}\prod_{j=1}^{n-1}\operatorname{Ch}\lambda_jx_j\int\limits_{\mathbb{R}^1_+}\operatorname{Cos}\lambda_nx_n\,d\nu_n(\lambda_1,\dots,\lambda_n).$$

Putting $x_1=\dots=x_{n-1}=0$ in every formula, we obtain

$$k(0,\dots,x_n)=\int\limits_{\mathbb{R}^1_+}\operatorname{Ch}\lambda_nx_n\int\limits_{\mathbb{R}^{n-1}_+}d\sigma_n(\lambda_1,\dots,\lambda_n)+\int\limits_{\mathbb{R}^1_+}\operatorname{Cos}\lambda_nx_n\int\limits_{\mathbb{R}^{n-1}_+}d\nu_n(\lambda_1,\dots,\lambda_n). \tag{2.5}$$

But since $k(x_1,\dots,x_n)$ is h.c., the second term in (2.5) must be equal to zero. In the result we have obtained the representation (2.4).

Now we show that the measures $\{\sigma_n(\cdot)\}$ are consistent. To do this, we shall consider these representations:

$$k(x_1,\dots,x_{n-1}) = \int\limits_{\mathbb{R}_+^{n-1}} \prod_{j=1}^{n-1} \operatorname{Ch} \lambda_j x_j \, d\sigma_{n-1}(\lambda_1,\dots,\lambda_{n-1}) \tag{2.6}$$

and

$$k(x_1,\dots,x_{n-1},x_n) = \int\limits_{\mathbb{R}_+^{n}} \prod_{j=1}^{n} \operatorname{Ch} \lambda_j x_j \, d\sigma_{n}(\lambda_1,\dots,\lambda_{n-1},\lambda_n). \tag{2.7}$$

Putting $x_n = 0$ in (2.7) we obtain

$$k(x_1,\dots,x_{n-1},0) = \int\limits_{\mathbb{R}_+^{n}} \prod_{j=1}^{n-1} \operatorname{Ch} \lambda_j x_j \, d\sigma_{n}(\lambda_1,\dots,\lambda_n). \tag{2.8}$$

From this follows the uniqueness of measures $d\sigma_{n-1}(\lambda_1,\dots,\lambda_{n-1})$ and $d\sigma_n(\lambda_1,\dots,\lambda_{n-1},\lambda_n)$, since $k(x_1,\dots,x_n) < ce^{\sum\limits_{j=1}^{n} N_j x_j^2}$. Then from (2.6) and (2.8) we shall obtain the condition of consistency of measures $\{d\sigma_n(\cdot)\}$:

$$\int\limits_{A\times\mathbb{R}_+^1} d\sigma_n(\lambda_1,\dots,\lambda_{n-1},\lambda_n) = \int\limits_{A} d\sigma_{n-1}(\lambda_1,\dots,\lambda_{n-1}),$$

where $A \subset \mathbb{R}_+^{n-1}$, A is Borel. But if a system of measures $\{\sigma_n(\cdot)\}$ is consistent, then we can construct the single measure $\sigma(\cdot)$ on $\mathbb{R}_+^\infty$ such that $\sigma(B \times \mathbb{R}_+^1 \times \mathbb{R}_+^1 \times \dots)$ for every $B \in \mathbb{R}_+^n$, B is Borel. Let X be a set of indices x arbitrary cardinality, and let $(R_x)_{x\in X}$ be a family of abstract spaces with σ-algebra $\mathcal{R}_x$ of subsets defined in each space. Assume that the measures on the spaces R_x are given. Now we want to construct a measure on the direct product $R_X = X_{x\in X} R_x$ of these spaces which, according to definition consists of all possible mappings of the form $X \ni x \to \lambda(x) \in R_x$. For arbitrary different points $x_1,\dots,x_n \in X$, let us denote $R_{x_1,\dots,x_n} = R_{x_1} \times \dots \times R_{x_n}$ $(n \in \mathbb{N})$. In $R_{x_1,\dots,x_n}$, we consider the σ-algebra $\mathcal{R}_{x_1,dots,x_n} = \mathcal{R}_{x_1} \times \mathcal{R}_{x_2} \times \dots \times \mathcal{R}_{x_n}$ of its subsets. The set $\mathbb{C} \in R_x$ is called cylindrical if it is determined by these points $x_1,\dots,x_n$ and base $\delta \in R_{x_1,\dots,x_n}$ according to the relation

$$\mathbb{C} = \mathbb{C}\left(x_1,\dots,x_n;\delta\right) = \left\{\lambda(\cdot) \in R_x \middle| \lambda(x_1),\dots,\lambda(x_n) \in \delta\right\}. \tag{2.9}$$

Assume that for any $x \in X$ some probability measure μ_x is given on the σ-algebra $\mathcal{R}_x$, i. e. $\mu_x\left(R_x\right) = 1$. There exists the standard Kolmogorov procedure, which enables us to construct a measure μ_x on the σ-algebra $\mathcal{C}_\sigma\left(R_x\right)$ from the family of measures $\left(\mu_x\right)_{x\in X}$; the measure μ_X is called a product of measures μ_x and is denoted by $\mu_X = \times_{x\in X}\mu_x$. Let us employ this procedure. Denote by $\mu_{x_1,\dots,x_n}$ $(n \in \mathbb{N})$ the measure on $\mathcal{B}_{x_1,\dots,x_n}$ obtained by the usual procedure of multiplying out a finite number of measures $\mu_{x_1,\dots,x_n} = \times_{k=1}^n \mu_{x_k}$. On cylindrical sets $\mathbb{C} \in \mathcal{C}\left(R_X\right)$, the measure μ_x is defined by

$$\mu_X(\mathbb{C}) = \mu_X\left(\mathbb{C}(x_1,\dots,x_n);\delta\right) = \mu_{x_1,\dots,x_n}(\delta). \tag{2.10}$$

The function of sets (2.10) satisfies the equalities $\mu_X(\varnothing) = 0$ and $\mu_X(R_X) = 1$ and is finitely additive. According to the classical theory of extension of measures (e.g. see Halmos [12], chapter 3) the finitely additive measure μ_X can be uniquely extended to the measure on the σ-algebra $\mathcal{C}\left(R_X\right)$. In the situation, which we are interested in the factor measures are given on the space $X_\kappa = \mathbb{R}^1_+$ with the Borel σ-algebra $\mathcal{B}_\kappa = \mathcal{B}(\mathbb{R}^1_+)$ $(\kappa \in \mathbb{N})$. The space $\mathbb{R}^\infty_+ = X^\infty_{\kappa=1}\mathbb{R}^1_+$ is equipped with the σ-algebra $\mathcal{C}_\delta\left(\mathbb{R}^\infty_+\right)$ generated by the cylindrical sets (2.9), which now have the form

$$\mathbb{C} = \mathbb{C}\left(1,\ldots,n;\delta\right) = \left\{\lambda \in \mathbb{R}^\infty_+ \,\middle|\, (\lambda_1,\ldots,\lambda_n) \in \delta \in \mathcal{B}\left(\mathbb{R}^\infty_+\right)\right\}.$$

Note that if $\mathbb{R}^\infty_+$ is considered as a topological space with the Tikhonov topology, then $\mathcal{C}_\delta\left(\mathbb{R}^\infty_+\right) = \mathcal{B}\left(\mathbb{R}^\infty_+\right)$.

So, as a result, we shall have such integral representation

$$k(x) = \int\limits_{\mathbb{R}^\infty_+} \prod_{j=1}^{\infty} \operatorname{Ch}\lambda_j x_j \, d\sigma(\lambda), \quad (x \in \mathbb{R}^\infty) \tag{2.11}$$

Let be prove now, that measure $\sigma(\times)$ concentrated on the cylindrical sets

$$\mathbb{C} = \left\{\lambda \in l^+_2 \,\middle|\, (\lambda_1,\ldots,\lambda_n) \in \mathcal{B}\left(l^{+,n}_2\right)\right\}$$

i.e. that $\sigma\left(l^+_2\right) = 1$.

To do this, we shall put

$$u(x) = u_{\mathrm{C}^\dagger}(x_1,\ldots,x_s) = \left(\frac{1}{2s}\right)^{\frac{s}{2}} \exp\left(-\frac{1}{2}\sum_{n=1}^{s} x_n^2\right) \in C^\infty_{\mathrm{C}^\dagger}\left(\mathbb{R}^s\right),$$

and

$$\widehat{u}(\lambda_1,\ldots,\lambda_s) = \int\limits_{\mathbb{R}^s} \prod_{n=1}^{s} \operatorname{Ch}\lambda_n x_n u(x)\, dx = e^{\sum\limits_{n=1}^{s}\frac{\lambda_n^2}{2}}.$$

Then, if we denote $v(x_1,\ldots,x_s) = u(x_1,\ldots,x_s)\sqrt{\pi}e^{x_1^2}\cdot\cdots\cdot\sqrt{\pi}e^{x_s^2}$, we shall get, on the basis of (2.11)

$$\int\limits_{\mathbb{R}^\infty}\int\limits_{\mathbb{R}^\infty} \frac{1}{2}\left[k(x+y)+k(x-y)\right]\overline{v(x)}v(y)\, d\omega_1(x)d\omega_1(y) = \lim_{d\to\infty}\int\limits_{\mathbb{R}^\infty_+}\left(\prod_{n=1}^{d}\frac{1}{2}\Big[\operatorname{Ch}\lambda_n\left(x_n+y_n\right)+\right.$$

$$\left.+\operatorname{Ch}\lambda_n\left(x_n-y_n\right)\Big],\overline{v(x)}v(y)\right)_{L_2(\mathbb{R}^\infty\times\mathbb{R}^\infty\times,d\omega_1(x)\otimes d\omega_1(y))} d\sigma(\lambda) =$$

$$= \int\limits_{\mathbb{R}^\infty_+} \exp\left(\sum_{n=1}^{s}\lambda_n^2\right) d\sigma(\lambda).$$

On the other hand, we have

$$\int_{\mathbb{R}^\infty}\int_{\mathbb{R}^\infty}\frac{1}{2}\left[k(x+y)+k(x-y)\right]\overline{v(x)}v(y)\,d\omega_1(x)d\omega_1(y)\le$$

$$\le\frac{c}{2}\int_{\mathbb{R}^\infty}\int_{\mathbb{R}^\infty}\left[e^{\sum\limits_{n=1}^{\infty}\lambda_n(x_n+y_n)^2}+e^{\sum\limits_{n=1}^{\infty}\lambda_n(x_n-y_n)^2}\right]\left|v(y)\overline{v(x)}\right|d\omega_1(x)d\omega_1(y)\le$$

$$\le c\int_{\mathbb{R}^\infty}\int_{\mathbb{R}^\infty}e^{\sum\limits_{n=1}^{\infty}2N_nx_n^2+2N_ny_n^2}\overline{|v(x)}v(y)|\,dx_1\dots dx_sdy_1\dots dy_s)d\omega_1(y)\cdot$$

$$\cdot\prod_{n=1}^{\infty}\left(\int_{\mathbb{R}^\infty}e^{2N_nx_n^2}\sqrt{\frac{1}{\pi}}e^{-x_n^2}\,dx_n\right)^2=c\prod_{n=1}^{s}\left(\int_{\mathbb{R}^1}e^{2N_nx_n^2}\sqrt{\frac{1}{2\pi}}e^{-\frac{x_n^2}{2}}\,dx_n\right)^2\cdot$$

$$\cdot\prod_{n=s+1}^{\infty}\frac{1}{1-2N_n}=c\prod_{n=1}^{s}\frac{1}{1-4N_n}\prod_{n=s+1}^{\infty}\frac{1}{1-2N_n}\le c\prod_{n=1}^{\infty}\frac{1}{1-4N_n}=c_1$$

$$\left(\text{as}\prod_{n=1}^{\infty}\frac{1}{1-4N_n}<\infty\right).$$

Thus, we have the estimate

$$\int_{\mathbb{R}_+^\infty}\exp\left(\sum_{n=1}^{s}\lambda_n^2\right)d\sigma(\lambda)<c_1\quad(s=1,2,\dots).\tag{2.12}$$

Since for any $\lambda\in\mathbb{R}_+^\infty$, $h(\lambda)=1\le\lim\limits_{s\to\infty}\exp\left(\sum\limits_{n=1}^{s}\lambda_n^2\right)\le\infty$, then by passing to a limit in (2.12) and taking into account the Fatou's lemma, we conclude that $h(\lambda)$ is sumable and therefore $d\sigma(\lambda)$ is almost everywhere $\lambda\in\mathbb{R}_+^\infty$, $h(\lambda)<\infty$, i.e. we show that

$$\sigma\left\{\lambda\in\mathbb{R}_+^\infty\middle|h(\lambda)=+\infty\right\}=0.$$

But $h(\lambda)$ exists if and only if when $\lambda\in l_2^+$.

That is why the representation (2.11) will look like this

$$k(x)=\int_{l_2^+}\prod_{j=1}^{\infty}\operatorname{Ch}\lambda_jx_j\,d\sigma(\lambda),\quad(x\in\mathbb{R}^\infty).\tag{2.13}$$

As $\left\|\prod\limits_{j=1}^{\infty}\operatorname{Ch}\lambda_jx_j\right\|_{L_2\left(\mathbb{R}^\infty;d\omega_{\frac{1}{2}}(x)\right)}<\infty$, if $\lambda\in l_2^+$, then the integral (2.13) converges strongly. Sufficiency is proved.

<u>Necessity</u> follows from the fact that $\left\|\prod\limits_{j=1}^{\infty}\operatorname{Ch}\lambda_jx_j\operatorname{Ch}\lambda_jy_j\right\|_{L_2(\mathbb{R}^\infty\times\mathbb{R}^\infty;d\omega_1(x)\otimes d\omega_1(y))}<\infty$ if $\lambda\in l_2^+$. Therefore from (2.13) we obtain the representation

$$\frac{1}{2}\left[k(x+y)-k(x-y)\right]=\int_{l_2^+}\prod_{j=1}^{\infty}\operatorname{Ch}\lambda_jx_j\operatorname{Ch}\lambda_jy_j\,d\sigma(\lambda),\quad(x\in\mathbb{R}^\infty).\tag{2.14}$$

With the help of (2.14) we check the inequality (2.2). Let us prove now the two last statements of the theorem. Let $u(x) = u_{C^\dagger}(x_1, \ldots, x_m)$, $u_{C^\dagger} \in C_0^\infty(\mathbb{R}^m)$, then with the help of (2.13), (2.14), (2.1) we obtain

$$\int\limits_{\mathbb{R}^\infty} \left(\int\limits_{l_2^+} \prod_{j=1}^{\infty} \operatorname{Ch} \lambda_j x_j \, d\sigma(\lambda) \right) \overline{u(x)} \, d\omega_{\frac{1}{2}}(x) = \lim_{n\to\infty} \int\limits_{l_2^+} \left(\int\limits_{\mathbb{R}^\infty} \prod_{j=1}^{n} \operatorname{Ch} \lambda_j x_j \right) \overline{u(x)} \, d\omega_{\frac{1}{2}}(x) d\sigma(\lambda) =$$

$$= \lim_{n\to\infty} \int\limits_{l_2^+} \left(\int\limits_{\mathbb{R}^\infty} \int\limits_{\mathbb{R}^\infty} \frac{1}{2} \left[\prod_{j=1}^{n} \operatorname{Ch} \lambda_j (x_j + y_j) \overline{u(x_j + y_j)} + \operatorname{Ch} \lambda_j (x_j - y_j) \overline{u(x_j - y_j)} \right] d\omega_1(x) \times \right.$$

$$\left. \times d\omega_1(y) \right) d\sigma(\lambda) = \int\limits_{\mathbb{R}^\infty} \int\limits_{\mathbb{R}^\infty} \frac{1}{2} \left[k(x+y) \overline{u(x+y)} + k(x-y) \overline{u(x-y)} \right] d\omega_1(x) d\omega_1(y) =$$

$$= \int\limits_{\mathbb{R}^\infty} k(x) \overline{u(x)} \, d\omega_{\frac{1}{2}}(x).$$

The validity of the equality (2.3) for $d\omega_{\frac{1}{2}}(x)$ for almost all $x \in \mathbb{R}^\infty$ follows from the arbitrariness of $u(x)$.

The uniqueness of measure $d\sigma(\lambda)$ follows from [1] (Theorem 3.9 Ch. VIII). The Theorem 1 is proved. $\square$

The Theorem 1 can be proven using the Theorem 2.4.1 from [10].

Since a kernel $\frac{1}{2}\left[k(x+y) + k(x-y)\right]$ is even by x, y, then we shall consider (2.2) on the even functions $u_n(x)$. Then

$$\begin{aligned} &\int\limits_{\mathbb{R}^\infty} \int\limits_{\mathbb{R}^\infty} \frac{1}{2} \left[k(x+y) + k(x-y) \right] \overline{u_n(x)} u_n(y) \, d\omega_1(x) d\omega_1(y) = \\ &= \int\limits_{\mathbb{R}^\infty} \int\limits_{\mathbb{R}^\infty} k(x+y) \overline{u_n(x)} u_n(y) \, d\omega_1(x) d\omega_1(y) \geq 0. \end{aligned} \tag{2.15}$$

Therefore, applying the theorem 2.4.1 from [10], we obtain the representation

$$k(x) = \int\limits_{l_2} e^{(\lambda, x)} \, d\rho(\lambda) = \int\limits_{l_2} \prod_{j=1}^{\infty} \operatorname{Ch} \lambda_j, x_j \, d\rho(\lambda), \tag{2.16}$$

where $d\rho(\lambda)$ is the non-negative even finite measure which is defined on the σ-algebra of cylindrical sets of l_2. The measure $\rho(\lambda)$ is even because the function class in (2.15) has changed. We shall show that the measure $\sigma(\lambda)$ in (2.11) has a support l_2^+: $\sigma(l_2^+) = 1$. For this purpose we shall go from the measure $\sigma(\cdot)$ to the even measure $\rho(\cdot)$, so that the projections ρ_n of measure $\rho(\cdot)$ will be determined by the projections $\sigma_n(\cdot)$ of measure $\sigma(\cdot)$. Then, since $\rho(l_2) = 1$ then $\sigma(l_2^+) = 1$ also, that is, we have the representation (2.3).

3 Hyperbolically convex functions on a nuclear space $\mathbb{R}_0^\infty$

Let $H_0 = l_2 = l_2(\mathbb{R}^1)$ be a space of square summable real sequences $l_2 \ni x = (x_\kappa)_{\kappa=1}^\infty$ with a scalar product $(x,y)_{H_0} = \sum\limits_{\kappa=1}^\infty x_\kappa y_\kappa$. Denote by T a set of all possible weights $\tau = (\tau_\kappa)_{\kappa=1}^\infty$, $\tau_\kappa \geq 1$, and put in correspondence with each $\tau \in T$ a Hilbert space

$$H_\tau = l_2(\tau) = \left\{ x \in l_2 \left| \sum_{\kappa=1}^\infty x_\kappa^2 \tau_\kappa = \|x\|_{H_\tau}^2 < \infty \right. \right\}$$

$$(x,y)_{H_\tau} = \sum_{\kappa=1}^\infty x_\kappa y_\kappa \tau_\kappa; \quad H_1 = H_0. \tag{3.1}$$

Evidently, $H_\tau \subset H_0$ topologically and $\|\cdot\|_{H_\tau} \geq \|\cdot\|_{H_0}$. The family of Hilbert spaces $(H_\tau)_{\tau\in T}$ is directed by imbedding, i.e. if for given $\tau' = (\tau'_\kappa)_{\kappa=1}^\infty \in T$ and $\tau'' = (\tau''_\kappa)_{\kappa=1}^\infty \in T$, we choose, for example, $\tau''' = (\tau''' = \tau' + \tau''_\kappa)_{\kappa=1}^\infty \in T$, then $H_{\tau'''} \subset H_{\tau'}$ and $H_{\tau'''} \subset H_{\tau'}$ topologically. Consider a space $\Phi = \operatorname{pr}\lim\limits_{\tau\in T} H_\tau$. This space is nuclear, since for every $\tau \in T$ one can take $\tau' = (2^\kappa \tau_\kappa)_{\kappa=1}^\infty$ such that the imbedding $O_{\tau',\tau} \colon H_{\tau'} \to H_\tau$ is quasinuclear. Indeed, let $(e_\kappa)_{\kappa=1}^\infty$ be a natural basis in l_2. Then the vectors $\left(\tau_\kappa^{-\frac{1}{2}} e_\kappa\right)$ from a basis in H_τ and therefore for the Hilbert norm of the imbedding operator $O_{\tau',\tau}$, we have

$$\|O_{\tau',\tau}\| = \sum_{\kappa=1}^\infty \left\| (\tau')^{-\frac{1}{2}} e_\kappa \right\|_{H_\tau}^2 = \sum_{\kappa=1}^\infty 2^{-\kappa} < \infty.$$

Obviously, the set Φ coincides with a collection of finite real sequence $\mathbb{R}_0^\infty$, i.e. $\mathbb{R}_0^\infty \ni \varphi = (\varphi_1, \dots, \varphi_n, 0, 0, \dots)$, where $n = n(\varphi)$ depends on a given sequence. This follows from the equality $\Phi = \bigcap\limits_{\tau\in T} l_2(\tau)$ and the fact that for a given sequence $\varphi = (\varphi_\kappa)_{\kappa=1}^\infty \in \Phi$, one can always take a weight $\tau \in T$, such that $\tau_\kappa = |\varphi_\kappa|^{-2} + 1$ provided that $\varphi_\kappa \neq 0$, and $\tau_\kappa = 1$ otherwise. Then the vector $\varphi \in H_\tau$ only in the case, when it has a finite number of nonzero coordinates.

For every $\tau \in T$, the Hilbert space $H_{-\tau} = l_2(\tau^{-1})$ is dual to $H_\tau = l_2(\tau)$ with respect to $H_\tau = l_2$. Here, $l_2(\tau^{-1})$ is constructed just as (2.10) by using the weight $\tau^{-1} = (\tau_\kappa^{-1})_{\kappa=1}^\infty$. According to the above argument the space Φ' coincides with $\bigcup\limits_{\tau\in T} H_{-\tau}$ of topology $\operatorname{ind}\lim\limits_{\tau\in T} H_\tau$. Hence, $\Phi' = \mathbb{R}^\infty$ ($\mathbb{R}^\infty$ is a set of all real sequence). In fact, for every vector $\xi = (\xi_\kappa)_{\kappa=1}^\infty \in \mathbb{R}^\infty$, let us take $\tau \in T : \tau_\kappa = (|\xi_\kappa| + 1)^2 \, 2^\kappa$ $(\kappa \in \mathbb{N})$. Then

$$\|\xi\|_{H_{-\tau}} = \sum_{\kappa=1}^\infty |\xi_\kappa|^2 (1 + |\xi|)^{-2} \, 2^{-\kappa} < \infty,$$

i.e. $\xi \in H_{-\tau}$. The scalar product in $H_0 = l_2$ defines a natural pairing of the elements of $\mathbb{R}_0^\infty$ and $\mathbb{R}^\infty$, namely,

$$(\xi, \varphi)_{H_0} = \sum_{\kappa=1}^\infty \xi_i \varphi_\kappa, \quad \left(\xi \in \mathbb{R}_0^\infty, \varphi \in \mathbb{R}^\infty\right).$$

Hence, we have constructed the nuclear rigging

$$\mathbb{R}^{\infty} \supset l_2\left(\tau^{-1}\right) \supset l_2 \supset l_2\left(\tau\right) \supset \mathbb{R}_0^{\infty}.$$

Thus we have $\mathbb{R}_0^{\infty} = \bigcap l_2\left(\{\tau_\kappa\}\right)$ is a nuclear linear topological space of real-valued finite sequences, which has a topology of the projective boundary of real Hilbert spaces $l_2\left(\{\tau_\kappa\}\right) = \left\{ t = (t_\kappa)_{\kappa=1}^{\infty} \middle| \|t\|^2_{l_2(\{\tau_\kappa\})} = \sum\limits_{\kappa=1}^{\infty} t_\kappa^2 \tau_\kappa < \infty, \tau_\kappa \geq 1, \kappa = 1, 2, \dots \right\}$ and $\mathbb{R}^{\infty} = \bigcup_{\tau_\kappa \geq 1} l_2\left(\{\frac{1}{\tau_\kappa}\}\right)$ is the space of all real-value sequences that have the topology of inductive limit of real Hilbert spaces $l_2\left(\{\frac{1}{\tau_\kappa}\}\right)$.

Now we consider one more topology, namely, the Mackey topology $\tau\left(\varphi', \varphi\right)$, which is defined as strongest topology on φ' consistent with the given duality between φ and φ' on the above-mentioned sense: One can show that for $\varphi = \operatorname{pr} \lim\limits_{\tau \in T} H$ the Mackey topology $\tau\left(\varphi', \varphi\right)$ admits a constructive description which coincides with the topology of the inductive limit $\operatorname{ind} \lim\limits_{\tau \in T} H_E$. Finally let us note that for a nuclear countable Hilbert space Φ the Mackey topology $\tau\left(\varphi', \varphi\right)$ coincides with the strong topology $\beta\left(\varphi', \varphi\right)$, which is given by the topology of the $\varphi = \operatorname{pr} \lim\limits_{\tau \in T} H_\tau$.

It follows that project and inductive topologies are coinciding.

The proof of the above statements on the consistency of topologies are given, e.g. in Schaefer ([13], chapter 4).

By parity for each variable we mean the function $k(\cdot)$, that satisfies equality

$$k\left(t_1, t_2, \dots, t_\kappa, \dots, t_n, 0 \dots\right) = k\left(t_1, t_2, \dots, -t_\kappa, \dots, t_n, 0 \dots\right)$$

$$\left(t \in \mathbb{R}^n \times (0, 0, \dots) \subset \mathbb{R}_0^{\infty}, n = 1, 2, \dots\right).$$

The function $k(t)$, which is even for the each variable on a nuclear space $\mathbb{R}_0^{\infty}$ is called *hyperbolically convex*, if it is even-positive defined and convex. That is for arbitrary $t^{(1)}, \dots, t^{(m)} \in \mathbb{R}_0^{\infty}$ and $\xi_1, \dots, \xi_m \in \mathbb{C}^1$ inequalities

$$\sum_{i,j=1}^{n} \frac{1}{2}\left[k\left(t^{(i)} + t^{(j)}\right) + k\left(t^{(i)} - t^{(j)}\right)\right] \xi_i \overline{\xi_j} \geq 0, \tag{3.2}$$

$$k\left(\frac{t^{(i)} + t^{(j)}}{2}\right) \leq \frac{1}{2}\left[k\left(t^{(i)}\right) + k\left(t^{(j)}\right)\right] \tag{3.3}$$

are holding. Suppose that for $k(t)$, if $t \in \mathbb{R}_0^{\infty}$ the estimate is true

$$|k(t)| \leq C e^{N\|t\|^2_{l_2(\{\tau_k\})}}, \quad (C > 0, N > 0), \tag{3.4}$$

then the following theorem is true:

Theorem 2. *In order that the function $k(t)$, which is given in the space $\mathbb{R}_0^{\infty}$ and satisfies the estimate* (3.4), *would allow such an integral representation*

$$k(t) = \int\limits_{\mathbb{R}_+^{\infty}} \prod_{\kappa=1}^{\infty} \operatorname{Ch} \lambda_\kappa t_\kappa \, d\sigma(\lambda), \quad \left(\lambda \in \mathbb{R}^{\infty}, \lambda_\kappa = (\lambda, e_\kappa)_{l_2}, t_\kappa = (t, e_\kappa)_{l_2}\right), \tag{3.5}$$

where $\rho(\cdot)$ is the non-negative, finite measure on the σ-algebra of cylindrical sets in $\mathbb{R}_+^\infty$ with Borel bases, it is necessary and sufficient that $k(t)$ be the e.p.d., convex and continuous on $\mathbb{R}_0^\infty$. The measures $\sigma(\cdot)$ for given $k(t)$ are defined uniquely.

Proof. Sufficiency. Let the function $k(t)$ be the e.p.d., convex and continuous on $\mathbb{R}_0^\infty$ and it satisfies the estimate (3.4). Let us prove that for $k(t)$ the integral representation (3.5) is true. Indeed, we restricted the continuous, e.p.d. function $k(t)$ on $\mathbb{R}^n$ to $\mathbb{R}_0^\infty$, which satisfies the estimate (3.4) and it is convex. For the function $k_n(t) = k(t_1, t_2, \ldots, t_n, 0, 0, \ldots)$ $(t \in \mathbb{R}^n)$ the following representation is true (2.4). The measures $\{\sigma_n(\cdot)\}$ are consistent. That is why due to the Kolmogorov's theorem it is possible to construct the single measure for the function $k(t)$ $(t \in \mathbb{R}_0^\infty)$. Hence, we have the integral representation (3.5). Sufficiency is proved.

Let us prove Necessity. Let the function $k(t)$ $(t \in \mathbb{R}_0^\infty)$ satisfies the condition (3.4) and has the representation (3.5). It's not hard to make sure that $k(t)$ is the e.p.d. and convex. Let prove now the continuity of $k(t)$, if $(t \in \mathbb{R}_0^\infty)$. It follows from

Lemma 1. *If the really-valued, e.p.d., convex function $k(c)$ $(c \in \mathbb{R}^1)$ admits the representation*

$$k(c) = \int_{\mathbb{R}_+^1} \operatorname{Ch} \lambda c \, d\sigma(\lambda),$$

where $\sigma(\lambda)$ is the finite measure on $\mathbb{R}_+^1$, then it is continuous.

.

Proof. Let $c_n \to c_0$ $(c_n, c_0 \in \mathbb{R}^1)$. It is necessary to prove that for an arbitrary $\varepsilon > 0$ exists such N that for $n \geq N$ the following inequality is true

$$\left| \int_{\mathbb{R}_+^1} \operatorname{Ch} \lambda c_n \, d\sigma(\lambda) - \int_{\mathbb{R}_+^1} \operatorname{Ch} \lambda c_0 \, d\sigma(\lambda) \right| < \varepsilon,$$

or that

$$\lim_{n\to\infty} \int_{\mathbb{R}_+^1} \left| \operatorname{Ch} \lambda c_n - \operatorname{Ch} \lambda c_0 \right| \, d\sigma(\lambda) = 0.$$

But according to the Lebesgue's theorem about the limit transition under the sign of integral we have

$$\lim_{n\to\infty} \int_{\mathbb{R}_+^1} \left| \operatorname{Ch} \lambda c_n - \operatorname{Ch} \lambda c_0 \right| \, d\sigma(\lambda) = \int_{\mathbb{R}_+^1} \lim_{n\to\infty} \left| \operatorname{Ch} \lambda c_n - \operatorname{Ch} \lambda c_0 \right| \, d\sigma(\lambda) = 0.$$

For the sequence of functions $f_n(\lambda) = |\operatorname{Ch} \lambda c_n - \operatorname{Ch} \lambda c_0|$ the major function will be the function $\varphi(\lambda) = 2\operatorname{Ch} \lambda c$. Therefore, $|f_n(\lambda)| = \varphi(\lambda)$ and

$$\int_{\mathbb{R}_+^1} \varphi(\lambda) \, d\sigma(\lambda) = 2 \int_{\mathbb{R}_+^1} \operatorname{Ch} \lambda c \, d\sigma(\lambda) = 2k(c) < \infty. \tag{3.6}$$

□

Now we prove the continuity of the e.p.d and convex function $k(t)$, if $t \in \mathbb{R}^n$.

Lemma 2. *If the function of n real variables $k(t_1, \dots, t_n) \in \mathbb{R}^n$ allows the representation*

$$k(t_1, \dots, t_n) = \int\limits_{\mathbb{R}_+^n} \prod_{\kappa=1}^{n} \operatorname{Ch} \lambda_\kappa t_\kappa \, d\sigma(\lambda_1, \dots, \lambda_n),$$

where $\sigma(\cdot)$ is the finite measure in $\mathbb{R}^n$, then it is continuous.

Proof. . Let $\left(t_1^{(j)}, \dots, t_n^{(j)}\right) \to \left(t_1^{(0)}, \dots, t_n^{(0)}\right)$ in $\mathbb{R}^n$. It is necessary to prove that for the any $\varepsilon > 0$ exists such N, that for every $j \geq N$

$$\left| \int\limits_{\mathbb{R}_+^n} \prod_{\kappa=1}^{n} \operatorname{Ch} \lambda_\kappa t_\kappa^{(j)} \, d\sigma(\lambda_1, \dots, \lambda_n) - \int\limits_{\mathbb{R}_+^n} \prod_{\kappa=1}^{n} \operatorname{Ch} \lambda_\kappa t_\kappa^{(0)} \, d\sigma(\lambda_1, \dots, \lambda_n) \right| \leq \varepsilon,$$

or that

$$\lim_{j \to \infty} \int\limits_{\mathbb{R}_+^n} \left| \prod_{\kappa=1}^{n} \operatorname{Ch} \lambda_\kappa t_\kappa^{(j)} - \prod_{\kappa=1}^{n} \operatorname{Ch} \lambda_\kappa t_\kappa^{(0)} \right| d\sigma(\lambda_1, \dots, \lambda_n) = 0.$$

But

$$\lim_{j \to \infty} \int\limits_{\mathbb{R}_+^n} \left| \prod_{\kappa=1}^{n} \operatorname{Ch} \lambda_\kappa t_\kappa^{(j)} - \prod_{\kappa=1}^{n} \operatorname{Ch} \lambda_\kappa t_\kappa^{(0)} \right| d\sigma(\lambda_1, \dots, \lambda_n) =$$

$$= \int\limits_{\mathbb{R}_+^n} \lim_{j \to \infty} \left| \prod_{\kappa=1}^{n} \operatorname{Ch} \lambda_\kappa t_\kappa^{(j)} - \prod_{\kappa=1}^{n} \operatorname{Ch} \lambda_\kappa t_\kappa^{(0)} \right| d\sigma(\lambda_1, \dots, \lambda_n) = 0.$$

The transition to the limit under the sign of integral is possible because according to the Lebesgue's theorem for the sequence of functions

$$f_j(\lambda_1, \dots, \lambda_n) = \left| \prod_{\kappa=1}^{n} \operatorname{Ch} \lambda_\kappa t_\kappa^{(j)} - \prod_{\kappa=1}^{n} \operatorname{Ch} \lambda_\kappa t_\kappa^{(0)} \right|$$

there is the major function $\varphi(\lambda) = \prod\limits_{\kappa=1}^{n} \varphi_\kappa(\lambda_\kappa)$, where

$$\varphi_\kappa(\lambda_\kappa) = 2\operatorname{Ch} \lambda_\kappa c_\kappa \quad \left(c_\kappa = \sup_j t_\kappa^2 \lambda_\kappa\right).$$

Therefore

$$|f_j \varphi(\lambda_1, \dots, \lambda_n)| \leq \varphi(\lambda_1, \dots, \lambda_n)$$

and

$$\int\limits_{\mathbb{R}_+^n} \prod_{\kappa=1}^{n} \varphi_\kappa(\lambda_\kappa) \, d\sigma(\lambda_1, \dots, \lambda_n) = 2^n k(c_1, \dots, c_n). \tag{3.7}$$

For $n = 1$, (3.6) follows from (3.7). The Lemma 2 is proved. □

Then the continuity of $k(t)$, if $t \in \mathbb{R}_0^\infty$, follows from the Lemma 2, as the continuity of $k(\cdot)$ in $\mathbb{R}_0^\infty$ is the continuity for every n functions $k_n(t_1, \ldots, t_n) = k(t)$ $(t \in \mathbb{R}^n)$, since the projective and inductive topologies in $\mathbb{R}_0^\infty$ are coinciding. The necessity is proved.

The uniqueness of measures in (3.5) follows from the uniqueness of measures $\rho_n(\lambda_1, \ldots, \lambda_n)$.$\blacksquare$

$\square$

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Лопотко О. В. *Інтегральне зображення парно додатно визначених обмежених функції нескінченного числа змінних* // Буковинський матем. журнал — 2023. — Т.11, №1. — С. 26–38.

Стаття складається з двох частин.

У першій частині доводиться інтегральне зображення для гіперболічно опуклих (г.о.) функцій $k(x)$ $\left(x \in \mathbb{R}^\infty = \mathbb{R}^1 \times \mathbb{R}^1 \times \ldots\right)$. Для цього в $\mathbb{R}^\infty$ вводимо міри $\omega_1(x)$, $\omega_{\frac{1}{2}}(x)$. Додатна визначеність (д.в.) для г.о. функцій розуміється в інтегральному сенсі відносно

міри $\omega_1(x)$. Далі ми доводимо, що міра $\rho(\lambda)$ в інтегральному зображенні для г.о. функцій зосереджена на $l_2^+ = \left\{\lambda \in \mathbb{R}_+^\infty = \mathbb{R}_+^1 \times \mathbb{R}_+^1 \times \ldots \,\middle|\, \sum\limits_{n=1}^{\infty} \lambda_n^2 < \infty\right\}$. Рівність для $k(x)$ $(x \in \mathbb{R}^\infty)$ розуміється майже всюди відносно міри $\omega_{\frac{1}{2}}(x)$.

У другій частині статті ми доводимо необхідну і достатню умови для інтегрального зображення г.о. функцій $k(x)$ $\big(x \in \mathbb{R}_0^\infty$ є ядерний простір$\big)$. Д.в. для г.о. функцій розуміється в точковому сенсі. Для цього потрібно сконструювати ланцюжок $\mathbb{R}_0^\infty \subset l_2 \subset \mathbb{R}^\infty$. Тоді, враховуючи, що проекційна та індуктивна топології співпадають, ми одержимо інтегральне зображення для г.о. функцій $k(x)$ $(x \in \mathbb{R}_0^\infty)$