DOI: https://doi.org/10.31861/bmj2025.02.09

Bandura A.I., Kryshtopa L.I., Mazur T.M., Ivasiv N.V., Skaskiv O.B.

GROWTH AND EXISTENCE OF ANALYTIC IN A BIDISC FUNCTIONS OF BOUNDED L-INDEX IN JOINT VARIABLES

We obtained growth estimates for bivariate functions which are analytic in the unit bidisc and have bounded **L**-index in joint variables. The positive continuous function $\mathbf{L}(z_1, z_2) = (l_1(z_2, z_2), l_2(z_1, z_2))$ satisfies additional behavior condition: for every point $z = (z_1, z_2)$ belonging the unit bidisc $\mathbb{D}^2$ the appropriate value of the function l_j at this point is greater than the reciprocal to $1-|z_j|$ multiplied by β, i.e. $l_j(z) > \beta/(1-|z_j|)$ for each $j \in \{1,2\}$ and some constant $\beta > 1$. Also we prove that for every analytic functions in the unit bidisc with bounded multiplcities of zero points there exists a positive continuous function $\mathbf{L}(z_1, z_2) = (l_1(z_2, z_2), l_2(z_1, z_2))$ providing boundedness of the **L**-index in joint variables for primary analytic function.

Key words and phrases: growth estimate, analytic function, bidisc, bounded **L**-index in joint variables, existence theorem, zero points with bounded multiplicities.

Ivano-Frankivsk National Technical University of Oil and Gas, 76019, 15 Karpatska street, Ivano-Frankivsk, Ukraine (Bandura A.I., Kryshtopa L.I., Mazur T.M.)
7 Boryslav School, 82300, 14 Volodymyr Velykyi Street, Boryslav, Ukraine (Ivasiv N.V.)
Ivan Franko National University of Lviv, 79000, 1 Universytetska Street, Lviv, Ukraine (Skaskiv O.B.)
e-mail: *andriykopanytsia@gmail.com (Bandura A.I.), l.i.kryshtopa@gmail.com (Kryshtopa L.I.), tetiana.mazur@nung.edu.ua (Mazur T.M.), petrechko.n@gmail.com (Ivasiv (Petrechko) N.V.), olskask@gmail.com (Skaskiv O.B.)*

1 Introduction

Theory of analytic functions in the unit bidisc having bounded **L**-index in joint variables was deeply developed by N.V. Petrechko with her co-authors A.I. Bandura and O.B. Skaskiv [2, 3, 6]. Moreover, there was proposed some application [11] of this theory to analytic solutions of linear higher order system of partial differential equations. But there are some early announced results at conferences in Chernivtsi University [9] which were not published with full proofs. This paper fills the indicated gap and presents existence theorem and growth estimates for this class of functions.

We consider two-dimensional complex space $\mathbb{C}^2$. Denote $\mathbb{R}_+ = [0, +\infty)$, $\mathbf{0} = (0,0) \in \mathbb{R}^2_+$, $\mathbf{1} = (1,1) \in \mathbb{R}^2_+$, $\mathbf{1}_1 = (1,0)$, $\mathbf{1}_2 = (0,1)$, $R = (r_1, r_2) \in \mathbb{R}^2_+$, $z = (z_1, z_2) \in \mathbb{C}^2$. For $A = (a_1, a_2) \in \mathbb{R}^2$, $B = (b_1, b_2) \in \mathbb{R}^2$ we will use formal notations without violation of the existence of these expressions

$$AB = (a_1b_1, a_2b_2), \quad A/B = (a_1/b_1, a_2/b_2),\ b_1 \neq 0,\ b_2 \neq 0, \quad A^B = a_1^{b_1} a_2^{b_2},\ b \in \mathbb{Z}^2_+,$$

УДК 517.55
2010 *Mathematics Subject Classification:* 32A10, 32A17, 32A22.

and the notation $A < B$ means that $a_j < b_j$, $j \in \{1,2\}$; the relation $A \le B$ is defined similarly. For $K = (k_1, k_2) \in \mathbb{Z}_+^2$ denote $\|K\| = k_1 + k_2$, $K! = k_1!k_2!$.

The bidisc $\{z \in \mathbb{C}^2 : \ |z_j - z_j^0| < r_j, \ j = 1,2\}$ is denoted by $\mathbb{D}^2(z^0, R)$ and the closed bidisc $\{z \in \mathbb{C}^2 : \ |z_j - z_j^0| \le r_j, \ j = 1,2\}$ is denoted by $\mathbb{D}^2[z^0, R]$, its skeleton is denoted by $\mathbb{T}^2(z^0, R) = \{z \in \mathbb{C}^2 : \ |z_1| = |z_2| = 1\}$, $\mathbb{D}^2 = \mathbb{D}^2(\mathbf{0}, \mathbf{1})$, $\mathbb{D} = \{z \in \mathbb{C} : \ |z| < 1\}$,where $z^0 = (z_1^0, z_2^0)$. For p, $q \in \mathbb{Z}_+$ and the partial derivative $\frac{\partial^{p+q} F(z_1,z_2)}{\partial z_1^p \partial z_2^q}$ of analytic function $F(z)$ in $\mathbb{D}^2$ we will use the notation

$$F^{(p,q)}(z) = F^{(p,q)}(z_1, z_2) := \frac{\partial^{p+q} F(z_1, z_2)}{\partial z_1^p \partial z_2^q}.$$

Let $\mathbf{L}(z) = (l_1(z), l_2(z))$, where $l_j(z) : \mathbb{D}^2 \to \mathbb{R}_+$ is a continuous function such that

$$(\forall z \in \mathbb{D}^2) \colon \ l_j(z) > \beta/(1 - |z_j|), \ j \in \{1,2\}, \tag{1}$$

and $\beta > 1$ is some constant. M.M. Sheremeta [14], V.O. Kushnir [10] imposed a similar condition for a function $l : \mathbb{D} \to \mathbb{R}_+$ and $l : G \to \mathbb{R}_+$, where G is arbitrary domain in $\mathbb{C}$. A.I. Bandura and O.B. Skaskiv [1] used the similar condition to consider analytic functions in the unit ball of bounded $\mathbf{L}$-index in joint variables.

An analytic function $F \colon \mathbb{D}^2 \to \mathbb{C}$ [2, 3, 6] is called a function of *bounded* $\mathbf{L}$*-index (in joint variables),* if there exists $n_0 \in \mathbb{Z}_+$ such that for all $z = (z_1, z_2) \in \mathbb{D}^2$ and for all $(p_1, p_2) \in \mathbb{Z}_+^2$

$$\frac{1}{p_1!p_2!} \frac{|F^{(p_1,p_2)}(z)|}{l_1^{p_1}(z) l_2^{p_2}(z)} \le \max\left\{ \frac{1}{k_1!k_2!} \frac{|F^{(k_1,k_2)}(z)|}{l_1^{k_1}(z) l_2^{k_2}(z)} : 0 \le k_1 + k_2 \le n_0 \right\}. \tag{2}$$

The least such integer n_0 is called the $\mathbf{L}$*-index in joint variables of the function* $F(z)$ and is denoted by $N(F, \mathbf{L}, \mathbb{D}^2) = n_0$.

We need some additional notations from [3]. By $Q(\mathbb{D}^2)$ we denote the class of functions $\mathbf{L}$, which satisfy condition (1) and the following characteristics are finite:

$$(\forall r_j \in [0, \beta], \ j \in \{1,2\}) \colon \ 0 < \lambda_{1,j}(R) \le \lambda_{2,j}(R) < \infty, \ \text{where}$$

$$\lambda_{1,j}(R) = \inf_{z^0 \in \mathbb{D}^2} \inf\left\{ \frac{l_j(z)}{l_j(z^0)} : z \in \mathbb{D}^2 \left[z^0, R/\mathbf{L}(z^0)\right] \right\}, \ \lambda_{2,j}(R) = \sup_{z^0 \in \mathbb{D}^2} \sup\left\{ \frac{l_j(z)}{l_j(z^0)} : z \in \mathbb{D}^2 \left[z^0, R/\mathbf{L}(z^0)\right] \right\}.$$

For an analytic function F in the unit bidisc we put $M(R, z^0, F) = \max\{|F(z)| \colon z \in \mathbb{T}^2(z^0, R)\}$. Then $M(R, z^0, F) = \max\{|F(z)| \colon z \in \mathbb{D}^2[z^0, R]\}$, because the maximum modulus for analytic function in a closed polydisc is attained on its skeleton.

We will use the following statement.

Theorem 1 ([3])**.** *Let* $\mathbf{L} \in Q(\mathbb{D}^2)$. *If an analytic function* F *in the unit bidisc* $\mathbb{D}^2$ *has bounded* $\mathbf{L}$*-index in joint variables, then for any* $R', R'' \in \mathbb{R}_+^2$, $\mathbf{0} < R' < R'' \le (\beta, \beta)$ *there exists* $p_1 = p_1(R', R'') \ge 1$ *such that for every point* $z^0 \in \mathbb{D}^2$ *one has*

$$M\left(R''/\mathbf{L}(z^0), z^0, F\right) \le p_1 M\left(R'/\mathbf{L}(z^0), z^0, F\right). \tag{3}$$

2 Growth of analytic in a bidisc functions of bounded L-index in joint variables

At first, we will prove the following lemma.

Lemma 1. *If $\mathbf{L} \in Q(\mathbb{D}^2)$, then for every $j \in \{1,2\}$ and for any fixed $z^* \in \mathbb{D}^2$ one has $|z_j| l_j(z^* + z_j\mathbf{1}_j) \to \infty$ as $|z_j^* + z_j| \to 1-0$.*

Proof. Taking into account (1), for $j \in \{1,2\}$ we obtain such an estimate

$$l_j(z^* + z_j\mathbf{1}_j) \geq \frac{\beta}{1-|z_j^*+z_j|} \to +\infty$$

as $|z_j^* + z_j| \to 1-0$. □

For growth estimates of analytic in a bidisc functions of bounded **L**-index in joint variables we suggest that $\mathbf{L}(z) = (l_1(|z_1|,|z_2|), l_2(|z_1|,|z_2|))$ and for all $z \in \mathbb{D}^2$, $j \in \{1,2\}$

$$l_j(|z_1|,|z_2|) > \frac{\beta}{1-|z_j|}, \tag{4}$$

where $\beta > 1$ is some constant. For simplicity we will denote $M(F,R) = \max\{|F(z)|\colon z \in \mathbb{T}^2(\mathbf{0},R)\}$, where $R = (r_1,r_2) < (1,1)$. Also the following notation $\boldsymbol{\beta} = (\beta,\beta)$ will be used. The growth estimates of various classes of holomorpic functions of several complex variables can be found [4, 5]. The following result was presented only in the dissertation of Petrechko N.V. [12].

Theorem 2. *Let $\mathbf{L} \in Q(\mathbb{B}^2)$ and $\mathbf{L}(z) = (l_1(|z_1|,|z_2|), l_2(|z_1|,|z_2|))$ satisfies (4). If an analyitc in the bidisc $\mathbb{D}^2$ function F has bounded $\mathbf{L}$-index in joint variables, then*

$$\ln M(F,R) = O\left(\min\left\{\int_0^{r_1} l_1(t,r_2)dt + \int_0^{r_2} l_2(r_1^0,t)dt; \int_0^{r_2} l_1(r_1,t)dt + \int_0^{r_2} l_2(t,r_2^0)dt\right\}\right)$$

as $r_1 \to 1-0$, $r_2 \to 1-0$, $R^0 = (r_1^0, r_2^0)$ are fixed radii.

Proof. Let $R > \mathbf{0}$, $R < \mathbf{1}$, and $z^* \in \mathbb{T}^2(\mathbf{0}, R + \frac{\boldsymbol{\beta}}{\mathbf{L}(R)})$ be a point such that

$$|F(z^*)| = \max\left\{|F(z)| : z \in \mathbb{T}^2\left(\mathbf{0}, R + \frac{\boldsymbol{\beta}}{\mathbf{L}(R)}\right)\right\}.$$

Let us denote $z^0 = \frac{z^* R}{R + \boldsymbol{\beta}/\mathbf{L}(R)}$. Then

$$|z_j^0 - z_j^*| = \left|\frac{z_j^* r_j}{r_j + \frac{\beta}{l_j(R)}} - z_j^*\right| = \left|\frac{z_j^* \beta/l_j(R)}{r_j + \frac{\beta}{l_j(R)}}\right| = \frac{\beta}{l_j(R)} \text{ and}$$
$$\mathbf{L}(z^0) = \mathbf{L}\left(\frac{z^* R}{R + \boldsymbol{\beta}/\mathbf{L}(R)}\right) = \mathbf{L}\left(\frac{(R + \boldsymbol{\beta}/\mathbf{L}(R))R}{R + \boldsymbol{\beta}/\mathbf{L}(R)}\right) = \mathbf{L}(R).$$

We consider two sceletons $\mathbb{T}^2(z^0, \frac{\mathbf{1}}{\mathbf{L}(z^0)})$ and $\mathbb{T}^2(z^0, \frac{\boldsymbol{\beta}}{\mathbf{L}(z^0)})$. By Theorem 1 there exists $p_1 = p_1(\mathbf{1}, \boldsymbol{\beta}) \geq 1$ such that (3) is valid for $R' = \mathbf{1}$, $R'' = \boldsymbol{\beta}$, i.e.

$$\max\left\{|F(z)|\colon z \in \mathbb{T}^2\big(\mathbf{0}, R + \frac{\boldsymbol{\beta}}{\mathbf{L}(R)}\big)\right\} = |F(z^*)| \leq \max\left\{|F(z)|\colon z \in \mathbb{T}^2\big(z^0, \frac{\boldsymbol{\beta}}{\mathbf{L}(R)}\big)\right\} =$$
$$= \max\left\{|F(z)|\colon z \in \mathbb{T}^2\left(z^0, \frac{\boldsymbol{\beta}}{\mathbf{L}(z^0)}\right)\right\} \leq p_1 \max\left\{|F(z)|\colon z \in \mathbb{T}^2\left(z^0, \frac{\mathbf{1}}{\mathbf{L}(z^0)}\right)\right\} \leq$$
$$\leq p_1 \max\left\{|F(z)| : z \in \mathbb{T}^2\left(\mathbf{0}, R + \frac{\mathbf{1}}{\mathbf{L}(R)}\right)\right\}. \tag{5}$$

The function $\ln^+ \max\{|F(z)|\colon z \in \mathbb{T}^2(\mathbf{0}, R)\}$ is a convex function in the variables $\ln r_1$, $\ln r_2$ (see [13, p. 84]). Then the function admits the representation

$$\ln^+ \max\{|F(z)|\colon z \in \mathbb{T}^2(\mathbf{0}, R)\} - \ln^+ \max\{|F(z)|\colon z \in \mathbb{T}^2(\mathbf{0}, R + (r_1^0 - r_1)\mathbf{1}_1)\} = \int_{r_1^0}^{r_1} \frac{A_1(t, r_2)}{t} dt, \tag{6}$$

$$\ln^+ \max\{|F(z)|\colon z \in \mathbb{T}^2(\mathbf{0}, R)\} - \ln^+ \max\{|F(z)|\colon z \in \mathbb{T}^2(\mathbf{0}, R + (r_2^0 - r_2)\mathbf{1}_2)\} = \int_{r_2^0}^{r_2} \frac{A_2(r_1, t)}{t} dt \tag{7}$$

for any $0 < r_j^0 \le r_j$, $j \in \{1, 2\}$, where the functions $A_1(t, r_2)$, $A_2(r_1, t)$ are positive non-decreasing functions in the variable t.

Using (5), we establish

$$\begin{gathered}\ln p_1 \ge \ln \max\left\{|F(z)|\colon z \in \mathbb{T}^2\left(\mathbf{0}, R + \frac{\boldsymbol{\beta}}{\mathbf{L}(R)}\right)\right\} - \ln \max\left\{|F(z)| : z \in \mathbb{T}^2\left(\mathbf{0}, R + \frac{\mathbf{1}}{\mathbf{L}(R)}\right)\right\} = \\ = \ln \max\left\{|F(z)|\colon z \in \mathbb{T}^2\left(\mathbf{0}, R + \frac{\boldsymbol{\beta}}{\mathbf{L}(R)}\right)\right\} - \ln \max\left\{|F(z)|\colon z \in \mathbb{T}^2\left(\mathbf{0}, R + \frac{\mathbf{1} + (\beta - 1)\mathbf{1}_2}{\mathbf{L}(R)}\right)\right\} + \\ \ln \max\left\{|F(z)|\colon z \in \mathbb{T}^2\left(\mathbf{0}, R + \frac{\mathbf{1} + (\beta - 1)\mathbf{1}_2}{\mathbf{L}(R)}\right)\right\} - \ln \max\left\{|F(z)| : z \in \mathbb{T}^2\left(\mathbf{0}, R + \frac{\mathbf{1}}{\mathbf{L}(R)}\right)\right\} = \\ = \int_{r_1 + 1/l_1(R)}^{r_1 + \beta/(l_1(R))} \frac{1}{t} A_1\left(t, r_2 + \frac{\beta}{l_2(R)}\right) dt + \int_{r_2 + 1/l_2(R)}^{r_2 + \beta/(l_2(R))} \frac{1}{t} A_2\left(r_1 + \frac{1}{l_1(R)}, t\right) dt \ge \\ \ge \ln\left(1 + \frac{\beta - 1}{r_1 l_1(R) + 1}\right) A_1\left(r_1, r_2 + \frac{\beta}{l_2(R)}\right) + \ln\left(1 + \frac{\beta - 1}{r_2 l_2(R) + 1}\right) A_2\left(r_1 + \frac{1}{l_1(R)}, r_2\right)\end{gathered} \tag{8}$$

By Lemma 1 one has $r_1 l_1(R) \to +\infty$ as $r_1 \to 1 - 0, r_2 \to 1 - 0$. Hence, for $j \in \{1, 2\}$ and $r_i \ge r_i^0$ we deduce

$$\ln\left(1 + \frac{\beta - 1}{r_j l_j(R) + 1}\right) \sim \frac{\beta - 1}{r_j l_j(R) + 1} \ge \frac{\beta - 1}{2 r_j l_j(R)}.$$

Thus, for every $j \in \{1, 2\}$ from inequality (8) it follows that

$$A_1\left(r_1, r_2 + \frac{\beta}{l_2(R)}\right) \le \frac{2 \ln p_1}{\beta - 1}\, r_1 l_1(R),$$
$$A_2\left(r_1 + \frac{1}{l_1(R)}, r_2\right) \le \frac{2 \ln p_1}{\beta - 1}\, r_2 l_2(R).$$

Let $R^0 = (r_1^0, r_2^0)$, where every r_j^0 is chosen above. Applying consequently equalities (6) and (7), we obtain

$$\begin{gathered}\ln \max\{|F(z)|\colon z \in \mathbb{T}^2(\mathbf{0}, R)\} = \\ = \ln \max\{|F(z)|\colon z \in \mathbb{T}^2(\mathbf{0}, R + (r_1^0 - r_1)\mathbf{1}_1)\} + \int_{r_1^0}^{r_1} \frac{A_1(t, r_2)}{t} dt = \\ = \ln \max\{|F(z)|\colon z \in \mathbb{T}^2(\mathbf{0}, R + (r_1^0 - r_1)\mathbf{1}_1 + (r_2^0 - r_2)\mathbf{1}_2)\} + \int_{r_1^0}^{r_1} \frac{A_1(t, r_2)}{t} dt + \int_{r_2^0}^{r_2} \frac{A_2(r_1^0, t)}{t} dt = \\ = \ln \max\{|F(z)|\colon z \in \mathbb{T}^2(\mathbf{0}, R^0)\} + \int_{r_1^0}^{r_1} \frac{A_1(t, r_2)}{t} dt + \int_{r_2^0}^{r_2} \frac{A_2(r_1^0, t)}{t} dt \le \\ \le \ln \max\{|F(z)|\colon z \in \mathbb{T}^2(\mathbf{0}, R^0)\} + \frac{2 \ln p_1}{\beta - 1}\Big(\int_{r_1^0}^{r_1} l_1(t, r_2) dt + \int_{r_2^0}^{r_2} l_2(r_1^0, t) dt\Big) \le\end{gathered}$$

$$\leq \ln\max\{|F(z)|\colon z\in\mathbb{T}^2(\mathbf{0},R^0)\}+\frac{2\ln p_1}{\beta-1}\Big(\int_0^{r_1} l_1(t,r_2)dt+\int_0^{r_2} l_2(r_1^0,t)dt\Big)\leq$$
$$\leq (1+o(1))\frac{2\ln p_1}{\beta-1}\Big(\int_0^{r_1} l_1(t,r_2)dt+\int_0^{r_2} l_2(r_1^0,t)dt\Big).$$

Hence, the following asymptotic equality holds

$$\ln\max\{|F(z)|\colon z\in\mathbb{T}^2(\mathbf{0},R)\}=O\Big(\int_0^{r_1} l_1(t,r_2)dt+\int_0^{r_2} l_2(r_1^0,t)dt\Big)$$

as $r_1\to 1-0$, $r_2\to 1-0$, Clearly, the similar equality can be proved for other permutation σ_2 of the set $\{1,2\}$, particularly,

$$\ln\max\{|F(z)|\colon z\in\mathbb{T}^2(\mathbf{0},R)\}=O\Big(\int_0^{r_2} l_1(r_1,t)dt+\int_0^{r_1} l_2(t,r_2^0)dt\Big)$$

Hence, Equation (2) is true. Theorem 2 is proved. □

3 Existence theorem for analytic functions in the unit bidisc with bounded multiplicities of zero points

Let Z_F be a zero set of the analytic function $F\colon \mathbb{D}^2\to\mathbb{C}$. If $z^0\in Z_F$, then by $p_F(z_1^0,z_2^0)$ we will denote the multiplicity of zero point $z^0=(z_1^0,z_2^0)$ of the function $F(z_1,z_2)$, i.e. for all $J=(j_1,j_2)\in\mathbb{Z}_+^2$ with $j_1+j_2<p_F(z^0)$ one has $F^{(j_1,j_2)}(z_1^0,z_2^0)=0$, but at least for one $J=(j_1,j_2)$ with $j_1+j_2=p_F(z_1^0,z_2^0)$ the following equality holds $F^{(j_1,j_2)}(z^0)\neq 0$.

The following theorem was obtained for entire functions of several complex variables in [8] and for functions analytic in the unit ball in [7]. For polydisc it was announced in [9].

Theorem 3. *In order that for an analytic function $F:\mathbb{D}^2\to\mathbb{C}$ there exist a positive continuous function $\mathbf{L}(z)=(l_1(z_1,z_2),l_2(z_1,z_2))$ satisfying* (1) *such that F is a function of bounded $\mathbf{L}$-index in joint variables it is necessary and sufficient that there exists $p\in\mathbb{Z}_+$ such that $p_F(z^0)\leq p$ for all $z^0\in Z_F$.*

Proof. The proof is similar to the proof of Theorem 4 in [7] for analytic functions in the unit ball.

Necessity. To simplify the notation we consider everywhere in the proof $p_0:=p_F(z_1^0,z_2^0)$. Necessity follows from the definition of bounded $\mathbf{L}$-index in joint variables. Indeed, assume on the contrary that

$$(\forall p\in\mathbb{Z}_+)(\exists(z_1^0,z_2^0)\in Z_F)\colon\quad p_0>p.$$

This means that for some $(j_1^0,j_2^0)\in\mathbb{Z}_+^2$ with $j_1^0+j_2^0=p_0$ one has $F^{(j_1^0,j_2^0)}(z_1^0,z_2^0)\neq 0$ and $F^{(j_1,j_2)}(z_1^0,z_2^0)=0$ for all $(j_1,j_2)\in\mathbb{Z}_+^2$ with $j_1+j_2\leq p_0-1$ and $(z_1^0,z_2^0)\in Z_F$. Therefore, the $\mathbf{L}$-index in joint variables at the point z^0 is not less than $p_0>p$, i.e.

$$N(F,\mathbf{L},z^0)\geq p_0>p.$$

If $p\to+\infty$, then we obtain that $N(F,\mathbf{L},z^0)\to+\infty$. This contradicts the boundedness of $\mathbf{L}$-index in joint variables of the function F.

Sufficiency. Let p be the least integer such that $\forall(z_1^0,z_2^0)\in Z_F$ one has $p_0\leq p$. Let $R=(r_1,r_2)\in\mathbb{R}_+^2$ with $r_j\in[0,1)$, $j\in\{1,2\}$. We define $R^0=(r_1^0,r_2^0)=\frac{1}{2}(\min\{1-r_1,r_1\},\min\{1-r_2,r_2\})$,

$K_R = \{z \in \mathbb{C}^2 \colon z \in \mathbb{D}^2[\mathbf{0}, R+R^0] \setminus \mathbb{D}^2(\mathbf{0}, R-R^0)\}$ for all $R=(r_1,r_2)\in[0,1)\times[0,1)$. Also we denote

$$m_1(r_1,r_2) = \min_{z^0\in K_R\cap Z_F} \max_{j_1+j_2\le p} \left\{ \frac{|F^{(j_1,j_2)}(z_1^0,z_2^0)|}{j_1!j_2!} : F^{(j_1,j_2)}(z_1^0,z_2^0)\neq 0 \right\}.$$

Since F is an analytic function in the unit bidisc $\mathbb{D}^2$, there exists

$$E = E(r_1,r_2) = (\varepsilon_1(r_1,r_2), \varepsilon_2(r_1,r_2)) > \mathbf{0}$$

such that

$$\frac{|F^{(j_1^0,j_2^0)}(z_1,z_2)|}{j_1^0!j_2^0!} \ge \frac{m_1(r_1,r_2)}{2}$$

for all

$$z\in K_R\cap G_E, \quad G_E = \bigcup_{z^0\in Z_F} \mathbb{D}^2(z^0, E(r_1,r_2)),$$

and for all (j_1^0,j_2^0) with $j_1^0+j_2^0\le p_0$ and $F^{(j_1^0,j_2^0)}(z_1^0,z_2^0)\neq 0$.

Let us denote

$$m_2(r_1,r_2) = \min\{|F(z)| \colon z\in\mathbb{D}^2[\mathbf{0},R+R^0], z\notin G_E\},$$
$$Q(r_1,r_2) = \min\{m_1(r_1,r_2)/2, m_2(r_1,r_2), 1\}.$$

For every $z=(z_1,z_2)\in\mathbb{D}^2$ we put $R=(|z_1|,|z_2|)$. Then at least one of the numbers $|F(z_1,z_2)|$, $\frac{|F^{(j_1,j_2)}(z_1,z_2)|}{j_1!j_2!}$ with $j_1+j_2\le p$ is not less than $Q(r_1,r_2)$ (respectively, $\frac{|F^{(j_1^0,j_2^0)}(z_1,z_2)|}{j_1^0!j_2^0!}$ for $(z_1,z_2)\in K_R\cap G_E$ and $|F(z_1,z_2)|$ for $(z_1,z_2)\in K_R\setminus G_E$). This yields

$$\max\left\{\frac{|F^{(j_1,j_2)}(z_1,z_2)|}{j_1!j_2!} : j_1+j_2\le p\right\} \ge Q(r_1,r_2). \tag{9}$$

Using Cauchy's formula for bidisc $\mathbb{D}^2[z,R^0]$ we deduce that for all $J=(j_1,j_2)$ with $j_1+j_2\ge p+1$ one has

$$\frac{|F^{(j_1,j_2)}(z_1,z_2)|}{j_1!j_2!} = \left|\frac{1}{(2\pi i)^2}\int_{\mathbb{T}^2(z,R^0)} \frac{F(\tau)}{(\tau-z)^{J+\mathbf{1}}}d\tau\right| \le \frac{1}{(r_1^0)^{j_1}(r_2^0)^{j_2}} \max\{|F(\tau)| \colon \tau\in\mathbb{D}^2[\mathbf{0},R+R^0]\}. \tag{10}$$

A positive continuous function $\mathbf{L}(z_1,z_2) = (l_1(z_1,z_2), l_2(z_1,z_2))$ can be chosen such that

$$l_1(z_1,z_2) = l_2(z_1,z_2) := \max\left\{\frac{\max\{1, \max\{|F(\tau)| \colon \tau\in\mathbb{D}^2[\mathbf{0},R+R^0]\}\}}{Q(r_1,r_2)(r_1^0r_2^0)^{1+\alpha}}, \frac{\beta}{1-r_1}, \frac{\beta}{1-r_2}\right\},$$

where $\beta>1$ is some constant, $R=(r_1,r_2)=(|z_1|,|z_2|)$ and $R^0=(r_1^0,r_2^0)=\frac{1}{2}(\min\{1-|z_1|,|z_1|\},\min\{1-|z_2|,|z_2|\})$, $\alpha\in(0,1)$ is fixed.

From (9) and (10) it follows that for all $j_1+j_2\ge\frac{p(1+\alpha)}{\alpha}$

$$\frac{|F^{(j_1,j_2)}(z_1,z_2)|/(j_1!j_2!l_1^{j_1}(z_1,z_2)l_2^{j_2}(z_1,z_2))}{\max\left\{\frac{|F^{(k_1,k_2)}(z_1,z_2)|}{k_1!k_2!l_1^{k_1}(z_1,z_2)l_2^{k_2}(z_1,z_2)} : k_1+k_2\le p\right\}} \le$$

$$\leq \frac{\max\{|F(\tau)|\colon \tau\in\mathbb{D}^2[\mathbf{0},R+R_0]\}}{Q(r_1,r_2)(r_1^0)^{j_1}(r_2^0)^{j_2}} \frac{1}{\max\{l_1^{j_1-k_1}(z_1,z_2)l_2^{j_2-k_2}(z_1,z_2): k_1+k_2\leq p\}} \leq$$
$$\leq \frac{\max\{|F(\tau)|\colon \tau\in\mathbb{D}^2[\mathbf{0},R+R_0]\}}{Q(r_1,r_2)((r_1^0r_2^0)^{1+\alpha})^{(j_1+j_2)/(1+\alpha)}} \min\{l_1^{k_1+k_2-j_1-j_2}(z_1,z_2): k_1+k_2\leq p\} \leq$$
$$\leq l_1^{p-j_1-j_2}(z)\Big(\frac{\max\{1,\max\{|F(\tau)|\colon \tau\in\mathbb{D}^2[\mathbf{0},R+R_0]\}\}}{Q(R)(r_1^0r_2^0)^{1+\alpha}}\Big)^{(j_1+j_2)/(1+\alpha)} =$$
$$= l_1^{p-(j_1+j_2)\alpha/(1+\alpha)}(z)\leq 1,$$

because $p-(j_1+j_2)\alpha/(1+\alpha)\leq 0$, $(j_1+j_2)/(1+\alpha)\geq \frac{p}{\alpha}\geq 1$ and $l_1(z_1,z_2)\geq 1$. Hence, we have that for all $j_1+j_2\geq \frac{p(1+\alpha)}{\alpha}$

$$\frac{|F^{(j_1,j_2)}(z_1,z_2)|}{j_1!j_2!l_1^{j_1}(z_1,z_2)l_2^{j_2}(z_1,z_2)} \leq \max\left\{\frac{|F^{(k_1,k_2)}(z_1,z_2)|}{k_1!k_2!l_1^{k_1}(z_1,z_2)l_2^{k_2}(z_1,z_2)}\colon\ k_1+k_2\leq p\right\}.$$

In view of arbitrariness of z, the analytic function F has bounded $\mathbf{L}$-index in joint variables and its $\mathbf{L}$-index in joint variables does not exceed $2p$, because $\frac{p(1+\alpha)}{\alpha}\to 2p$ as $\alpha\to 1$. □

REFERENCES

[1] Bandura A., Skaskiv O. *Functions analytic in a unit ball of bounded* **L***-index in joint variables*. J. Math. Sci. 2017, **227** (1), 1–12. doi:10.1007/s10958-017-3570-6

[2] Bandura A.I., Petrechko N.V., Skaskiv O.B. *Analytic functions in a polydisc of bounded* **L***-index in joint variables*. Mat. Stud. 2016, **46** (1), 72–80. doi:10.15330/ms.46.1.72-80

[3] Bandura A., Petrechko N., Skaskiv O. *Maximum modulus in a bidisc of analytic functions of bounded* **L***-index and an analogue of Hayman's theorem*. Mat. Bohemica 2018, **143** (4), 339–354. doi:10.21136/MB.2017.0110-16

[4] Bandura A.I., Skaskiv O.B. *Analytic functions in the unit ball of bounded L-index: asymptotic and local properties*. Mat. Stud. 2017, **48** (1), 37–73. doi:10.15330/ms.48.1.37-73

[5] Bandura A., Skaskiv O. *Asymptotic estimates of entire functions of bounded* **L***-index in joint variables*. Novi Sad J. Math. 2018, **48** (1), 103–116. doi: 10.30755/NSJOM.06997

[6] Bandura A.I., Petrechko N.V. *Properties of power series of analytic in a bidisc functions of bounded* **L***-index in joint variables*. Carpathian Math. Publ. 2017, **9** (1), 6–12. doi:10.15330/cmp.9.1.6-12

[7] Bandura A., Skaskiv O. *Analytic functions in the unit ball of bounded* **L***-index in joint variables and of bounded L-index in direction: a connection between these classes*. Demonstr. Math. 2019, **52** (1), 82–87. doi: 10.1515/dema-2019-0008

[8] Bandura A.I., Skaskiv O.B. *Iyer's metric space, existence theorem and entire functions of bounded* **L***-index in joint variables*. Bukovin. Mat. J. 2017, **5** (3-4), 8–14. (in Ukrainian)

[9] Bandura A., Petrehcko N. *Existence theorem for analytic in the polydisc function of bounded* **L***-index in joint variables*. In: Conference materials "Modern problems of mathematics and its application in natural sciences and information technologies", dedicated

to the 50th anniversary of the Faculty of Mathematics and Informatics at Yuriy Fedkovych Chernivtsi National University. September 17-19, 2018, Chernivtsi, Ukraine. 164–165. (in Ukrainian) `http://fmi50.pp.ua/files/FMI50-Materials.pdf?i=1`

[10] Kushnir V.O., Sheremeta M.M. *Analytic functions of bounded l-index*, Mat. Stud. 1999, **12** (1), 59–66.

[11] Petrechko N. *Bounded* $\mathbf{L}$*-index in joint variables and analytic solutions of some systems of PDE's in bidisc.* Visn. Lviv Univ. Ser. Mech. Math. 2017, **83**, 100–108.

[12] Petrechko N.V. *Properties of functions of bounded index in the unit bidisc.* Diss. Cand. Phys.-Math. Sci. Lviv, 2019. (in Ukrainian) `https://lnu.edu.ua/wp-content/uploads/2019/04/dys_petrechko.pdf`

[13] Ronkin L.I. *Introduction to the theory of entire functions of several variables*, 1 edn. AMS, Providence (1974)

[14] Sheremeta M., *Analytic functions of bounded index.* VNTL Publishers, Lviv (1999).

Received 16.10.2025

Бандура А.І., Кріштопа Л.І., Мазур Т.М., Івасів Н.В., Скасків О.Б. *Зростання та існування аналітичних у бікрузі функцій обмеженого* $\mathbf{L}$*-індексу за сукупністю змінних* // Буковинський матем. журнал — 2025. — Т.13, №2. — С. 80–87.

Отримано оцінки зростання для аналітичних в одиничному бікрузі функцій обмеженого $\mathbf{L}$-індексу за сукупністю змінних. Відповідна додатна неперервна функція $\mathbf{L}(z_1, z_2) = (l_1(z_2, z_2), l_2(z_1, z_2))$ задовольняє додаткову умову на поводження в бікрузі: для кожної точки $z = (z_1, z_2)$ з одиничного бікруга $\mathbb{D}^2$ відповідне значення функції l_j у цій точці більше за величину, обернену до $1 - |z_j|$, помножену на β, тобто, $l_j(z) > \beta/(1 - |z_j|)$ для кожного $j \in \{1, 2\}$ та деякої сталої $\beta > 1$. Також ми доводимо, що для кожної аналітичної в одиничному бікрузі функції з обмеженою кратністю нульових точок існує додатна неперервна функція $\mathbf{L}(z_1, z_2) = (l_1(z_2, z_2), l_2(z_1, z_2))$, для якої така аналітична функція матиме обмежений $\mathbf{L}$-індекс за сукупністю змінних.